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ENCEPHALITIS AND POLIOMYELITIS

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In 1916 there arose in Vienna and possibly in other places in the Austrian Empire, cases of a peculiar disease a prominent symptom of which is lethargy. Because of this feature, which amounts at times to a condition of profound and prolonged "sleepiness," the disease receives the popular name of sleeping sickness. And yet it was quickly recognized as being wholly distinct from the well-known African sleeping sickness, the inciting microörganism of which is a trypanosome.

The structural or histologic basis of the Austrian disease is an inflammation, chiefly located in the gray matter of the base of the brain, or technically an encephalitis. Because of the peculiar symptoms and of the lesions, von Economo of Vienna who first described the disease gave it the name of lethargic encephalitis.

About two years later, that is, in 1918, outbreaks of the same disease were reported from England and France. A little later the disease appeared in the United States, and published reports now indicate that it has a wide distribution over the world.

The total number of cases which have been observed and reported now run into the hundreds. Indeed, some writers prefer to call the disease "epidemic encephalitis." Up to the present time no great epidemic outbreak has been reported. However, the conditions with reference to the disease are more or less disquieting.

It is interesting to note that when the disease was first recognized in Vienna, it was supposed to be a form of food (sausage) poisoning; and the early cases in England were attributed to botulism. This mistake is quite explicable: among the prominent symptoms are paralyses of the ocular and facial muscles, which arise also in food poisoning. Because of war conditions, both the countries mentioned were using preserved foods to an unusual degree; hence the inference.

Because of the accompanying paralysis and subsequently because of certain peculiar characteristics of the histological changes in the brain, a discussion arose whether the disease was not merely a peculiar form of poliomyelitis or infantile paralysis, which has prevailed epidemically and fitfully in Europe and America for a decade or longer.

It is on this last point that I wish to make a few remarks. At the Rockefeller Institute we have had what may be regarded as a wide experience with poliomyelitis during ten or twelve years; and since the winter of 1919 and up to the present time we have had referred to us material from numbers of cases of the form of encephalitis we are considering.

On the basis of the studies we have carried out with the latter, it is quite safe to say that the diseases—lethargic or epidemic encephalitis and epidemic poliomyelitis—are distinct. A striking difference relates to the transmissibility of the one to lower animals (monkeys) and not of the other. No special difficulty is encountered in communicating poliomyelitis from man to monkey; we have not succeeded in transferring the encephalitis to animals. Moreover, the actual histological changes or lesions in the two diseases show striking differences, both of distribution in the central nervous organs and in minute structure. Lethargic encephalitis is surely an infectious disease, but its inciting microbe is still undiscovered.

The question arises whether encephalitis has been observed before, or is to be considered as being a new disease, in the sense that it had never before been recognized and described. In this connection it may be said that a brief account exists of an outbreak of a so-called "sleeping sickness" in the neighborhood of Tübingen in Germany in the year 1712; and what appears to be a similar malady more widely distributed in Austria, Italy, and Switzerland in 1890. There is a certain amount of mystery attached to the latter outbreak, partly because of the odd name "nona" it received, which has not been satisfactorily traced etymologically. It has been suggested that the term is a mere corruption of "coma," and partly because the chief accounts of it are given in the lay rather than the medical press of the time.

It is, however, of some significance that the first appearance of the present epidemic and the location of the 1890 outbreak coincide territorially. It is, of course, not absolutely possible to state that the disease originated this time in Vienna; under war conditions its occurrence in outlying rural districts might well have been overlooked. But there is nevertheless basis for the view that I shall propose tentatively, namely, that an endemic focus of a peculiar and specific form of encephalitis exists in Southeastern Europe, from which the present wide occurrence of the disease took origin. Should this view come to be established, we shall have learned of a new and previously unsuspected focus of an endemic-epidemic disease of world importance.

A STATISTICAL METHOD FOR STUDYING THE RADIATIONS FROM RADIOACTIVE SUBSTANCES AND THE X-RAYS AND ITS APPLICATION TO SOME γ-RAY PROBLEMS

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Statistical methods of studying X-rays and the radiations from radioactive substances are important partly because they make it possible, to some extent at least, to investigate the individual rays. The scintillation method of counting α-particles for example brought decisive evidence of the atomic structure of matter. It also made it possible to obtain directly the charge on an α-particle. Furthermore, the experiments on the scattering of α-particles gave Rutherford some definite knowledge which led to some of the most fruitful suggestions about atomic structure.

The electrical method of counting α-particles as used by Rutherford and Geiger was adapted by Geiger,[1] after some changes, to the statistical study of both α-particles and β-particles. Kovarik and McKeehan[2] applied the method to the investigation of the absorption and scattering of β-particles getting direct information about the β-particles, whereas the previous information was all deduced from ionization experiments which to be accurate demands a complete knowledge of the variation of the speed and of ionization with speed of a β-particle passing through matter. By this method they also obtained evidence[3] supporting the single scattering theory of Rutherford and got results in agreement with those deduced from Wilson's photographs of the tracks of β-particles produced by the condensation method—another important statistical method. They also investigated the magnetic spectra[4] of the β-particles from radium B and C and from radium D and E. In their first report they called attention to the fact that the γ-ray effects were also counted and that correction had to be made for these.

A more recent development[5] in the method makes it possible for each ray, upon entering the counting chamber, to cause an audible sound or to make an automatic record on paper. The counting chamber generally surrounds a specially prepared point. The chamber and the point are at a high difference of potential—near the sparking potential. When any of the rays enter the counting chamber, ionization is produced and on account of the intense field between the point and the chamber, ionization by collision results. This magnifies the effect initially produced by the ray and this is further magnified by an audion bulb operating a sensitive relay which operates a local circuit by means of which a chronograph pen makes a record of the fact that an α-particle, a β-particle, a γ-ray pulse or an X-ray pulse has entered the counting chamber.

Using this method it is possible to investigate some important problems

pertaining to the nature of γ-rays and X-rays, and also questions bearing on energy absorption and emission by an atom. For example it is important to know whether a spherical wave passing over matter will cause a simultaneous ejection of electrons at different parts of the same wave front. To test this two independent counting chambers were placed at equal distances from the source of γ-rays but at various different relative positions. One chamber registered the γ-ray effect audibly and the other optically. It was found that in no case were the effects produced simultaneously. When the chambers were placed one behind the other the effects were also non-simultaneous.

By varying the distance between one chamber and the source a confirmation of the inverse square law was obtained. Using a chamber of some definite material with thin walls and increasing this small thickness of the walls the registered number of effects increased. When material of higher atomic weight was used the effect also increased.

The first experiments mentioned can best be explained by assuming some corpuscular or ether-string theory and applying the law of probability to the distribution of the radiations, but in view of the facts coming from crystal experiments it seems necessary to consider some spreading pulse theory. If we assume the pulses to spread out as spherical surfaces we must consider absorption of energy by an atom and also a possible trigger action—an action not in great favor in recent years. The fact that the counting chambers did not record the γ-ray effects simultaneously means that the electron or electrons ejected were not ejected simultaneously from the metal or air of the chamber; but since the electrons ejected have, presumably, equal velocities and, therefore, equal energies, the experimental result indicates that the individual electrons do not acquire the necessary quanta of energy at the same time. This suggests a possible storing up of energy. There are, however, reasons to believe that the energy stored up cannot be equal to the energy represented by the ejected electrons, and it would follow that some if not most of the energy of the ejected electrons must come from the interior of the atom and that the γ-ray energy absorbed is only sufficient to perform the trigger action. Furthermore, it seems very probable that the particular electron to be ejected may have previously acquired, in some way, energy nearly sufficient to throw it out of balance which the γ-ray energy may suffice in doing; and it also seems probable that the more penetrating the γ-rays, i. e., the higher the value of ν, the deeper is the electron which will be ejected by the γ-ray.

During the progress of these experiments, which were to include also the determination of the number of γ-ray pulses per second, per gram of radium, I received reprints of work done on this problem by R. W. Lawson and V. F. Hess[6] at the Radium Institute in Vienna (where Mr. Lawson, although a civilian prisoner during the war, was permitted to continue

his work). Their counting chamber was a Rutherford and Geiger's spherical chamber with gas at reduced pressure. They obtained 2.92×10^{10} γ-rays from radium B and C per second per gram of radium. The material of their counting chamber was copper. In view of my results with chambers of different materials I decided to continue my experiments on this point especially as my results indicated that Lawson and Hess's value is probably low. My results at present indicate a value of 7×10^{10} γ-rays from radium B and C per second per gram of radium.

[1] Geiger, H., *Verh. D. Phys. Ges.*, **15**, 1913 (534).
[2] Kovarik, A. F., and McKeehan, L. W., *Physik. Z.*, **15**, 1914 (434).
[3] Kovarik, A. F., and McKeehan, L. W., *Physic. Rev.*, N. S., **6**, 1915 (426).
[4] Kovarik, A. F., and McKeehan, L. W., *Ibid.*, **8**, 1916 (574).
[5] Kovarik, A. F., and McKeehan, L. W., *Ibid.*, **13**, 1919 (272).
[6] Hess, V. F., and Lawson, R. W., *Sitzb. Akad. Wien*, **125**, 1916 (585).

ON THE NATURE OF THE HEAT PRODUCTION IN A SYSTEM OF PLATINUM BLACK, ALCOHOL, AND AIR

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Communicated by R. A. Millikan, January 16, 1920

When a small quantity of platinum black is introduced into an atmosphere of air saturated with ethyl or methyl alcohol vapor a violent reaction takes place on the surface of the platinum causing it to become incandescent.

The nature of this reaction has been explained in two ways. It is held by some that owing to the vast surface and poor heat conductivity of platinum black the adsorption of the alcohol vapor liberates sufficient heat to raise the temperature of the platinum considerably. This rise in temperature is supposed to accelerate the rate of oxidation of the alcohol vapor in the neighborhood of the platinum thus further raising its temperature until the substance becomes incandescent.*

On the other hand it is held by others that the heat produced at the surface of the platinum black is due primarily to the oxidation of the alcohol at the surface of the platinum. The adsorption of the alcohol is in this case considered as contributing relatively little heat. The surface of the platinum is, however, supposed to exert a "catalytic" action on the alcohol vapor and oxygen increasing the speed of reaction to many times that in air at ordinary temperatures.

The writer's attention was called to the two explanations by Prof. Millikan and it occurred to the writer that a decision between the two explanations might easily be obtained experimentally. If a thermocouple

* This explanation is ascribed to Wilhelm Ostwald by Prof. P. Sabatier, in a course of lectures given students of the A. E. F., at Toulouse, France, in 1918.

be covered with platinum black it is an easy matter to determine the relative rise in temperature of the platinum black due to the adsorption of alcohol vapor in the presence and in the absence of air. Such experiments were performed by the writer. As the results were so decisive and as no reference to similar experiments could be found in the literature, a brief statement of the observations may be of interest.

About 0.41 gram of platinum black, prepared from $PtCl_4$ by precipitation with aluminium from an acid solution, was placed in a small beaker in a vacuum desiccator. The base of the desiccator contained a small quantity of H_2SO_4 which served to dry the platinum black. The beaker was surrounded as completely as possible with cotton wool to cut down convection currents. The junction of a "Chromel-Alumel" thermocouple of No. 18 wire welded by copper was buried in the heap of platinum black. The leads passed through a rubber stopper to the circuit of a galvanometer which was closed through a key.

The sensibility of the galvanometer used had several values lying between a deflection of 1 cm. scale reading for 4.9° C. change in temperature to 1 cm. scale reading for 0.36° C. change in temperature. Greater sensibility could not be used as the temperature changes due to compression and exhaustion of the air in the experiments were in the neighborhood of one degree. By means of a two-way stopcock the desiccator could alternately be exhausted by an air pump or filled with a mixture of CO_2 and CH_3OH, a mixture of air and CH_3OH, or by CO_2 alone. The gases were saturated with CH_3OH by bubbling through a small quantity of the liquid in the bottom of a bottle.

The following is the record of a typical experiment. After having been exposed to air and alcohol, the chamber was exhausted to about 2 cm. Hg pressure and washed out with CO_2 gas at least three times. The galvanometer deflection after each filling registered about 4 cm. corresponding to a rise of about 1.44° C. This was due to the heat of compression of the CO_2 on filling, as the pressure of the CO_2 in the generator was above that of the air. Exhaustion of the gas in the desiccator also caused a drop of temperature of about 0.4° C.

Alcohol vapor plus CO_2 was then admitted and the deflection amounted to 5.5 cm. or to about 1.98°. The rise due to adsorption of the alcohol, or due to *adsorption* and *oxidation* of adsorbed alcohol by the remnants of the adsorbed O_2 left on the platinum, therefore, amounted to about 0.54° C. The mean of a number of results gave the value of the rise produced due to adsorption as about 0.36° C.

If the gas admitted after exhaustion was a mixture of air and CH_3OH vapor, instead of CO_2 and CH_3OH, the resulting deflection was on the average 24 cm. at the end of about a minute. The sensibility of the galvanometer was less than before so that this corresponded to a rise of 34° C. Different sensibilities gave about the same value for the rise in tempera-

ture. In this case the deflection was not instantaneous but the temperature rose more gradually, reaching a maximum in about 45 seconds. After this it slowly fell as the alcohol and air were consumed in the immediate neighborhood of the platinum.

When air alone was admitted to the apparatus no appreciable rise in temperature was detected.

It is obvious that the heat liberated on the surface of platinum black due to adsorption alone either of air or of alcohol vapor is small compared to that liberated in the process of oxidation of the alcohol at the surface of the platinum. In fact, adsorption gives at most only 1% of the heat produced in these experiments.

It will be of interest to know whether this conclusion is also supported by other facts.

On the basis of a recent theory of Langmuir[1] that the layer of molecules of a gas *adsorbed* on a solid surface is never more than one molecule thick, it is possible with certain assumptions to calculate the rise in temperature of the platinum black. If one considers that the heat of adsorption is equal to the heat of condensation, if one neglects heat losses due to radiation, conduction, and convection, and if one assumes a definite radius for the particles of the platinum black the calculation is quite simple.

Let us assume that the platinum black is made up of aggregates of spheres of platinum whose diameter is 100 $\mu\,\mu$, or 1×10^{-4} cm. This is about the size of the colloidal gold particles of a gold suspension investigated by Zsigmondy.[2] Such an assumption is justifiable as the particles of platinum black used are certainly much larger than this.

Using 0.41 gram of platinum black we get

The number particles of diameter 1×10^{-4} cm. in 0.41 gram platinum black $= 3.75 \times 10^{10}$.

The total surface of all the particles $= 1.18 \times 10^{3}$ sq. cm.

On Langmuir's theory there should be $= 0.85 \times 10^{15}$ molecules of CH_3OH per square cm. surface in an adsorbed layer one molecule thick.

Total number of molecules of CH_3OH condensed on the surface of the platinum black $= 1 \times 10^{18}$ molecules.

Heat generated by these molecules on condensation equals heat of evaporation of the 10^{18} molecules $= 1.53 \times 10^{-2}$ calories.

The 0.41 gram of platinum will be raised 1.16° C. at 0° C. by this adsorption.

If the amount adsorbed were oxidized the temperature would be raised 19.8° C.

It is obvious that the relative values calculated on these assumptions are in approximate agreement with the values experimentally obtained. The surface of the platinum black is in reality much smaller than was assumed here. Furthermore the heat liberated by oxidation is continuously liberated as long as alcohol and oxygen are present, while in the case

of adsorption the heat is all liberated in the short span of time during which the adsorbed layer is forming. It is therefore obvious that in spite of the fact that the heat of adsorption may be greater than that of condensation the rise in temperature observed would be mainly due to the oxidation of the alcohol as was found by experiment to be the case.

In view of what has preceded it is to be concluded that the rise in temperature of the platinum black due to adsorption of the alcohol vapor is not the initiator of the reactions which heat the platinum to incandescence. For the heat generated by adsorption is negligible compared to that produced by oxidation and would not be capable of increasing the speed of reaction appreciably.

It is then on the basis of an increased rate of oxidation of the alcoho at the surface of the platinum ("catalytic" action) coupled with the physical properties of the platinum black that one must explain the rise in temperature. In this, one is justified to some extent, for it has long been known that surfaces of platinum accelerate certain chemical reactions, both in gases and in solution. Recently, Langmuir[1] has shown that at ordinary temperatures and at reduced pressures platinum foil when "activated" will cause oxygen and carbon monoxide to unite. In this case there seems to be some close relation between this action of platinum and adsorption, but it is not due to any heating effect.

[1] Langmuir, I., *J. Amer. Chem. Soc., Easton, Pa.*, **40**, 1918 (1361).
[2] Zsigmondy, R., *Zur Erkenntnis der Kolloide*, Jena, 1905 (104).

LEPTOSPIRA ICTEROIDES AND YELLOW FEVER

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Although yellow fever is being rapidly controlled by measures directed against the intermediate host (the Stegomyia mosquito), which acts as the vector, yet the nature of its inciting microbe has remained unknown. Hence yellow fever is a striking instance of the fact that given precise knowledge of the mode of infection of a microbic disease effective practical measures may be evolved for its control, even though the inciting microbe remain undiscovered.

Recently a fresh opportunity was afforded for the investigation of the etiology of yellow fever. The International Health Board of the Rockefeller Foundation sent a commission to Guayaquil, Ecuador, in June, 1918, to survey the field preparatory to the application of measures of control. Guayaquil has long been a recognized endemic focus of yellow fever, from which periodic extension has taken place to Central America.

I was attached to this commission as bacteriologist, and my attention was particularly directed, although not wholly confined to, the possibility that the inciting microbe belonged to the group of spiral microorganisms which in the last ten or twelve years have come to be regarded as playing a very important part in human pathology. There was already known, indeed, a particular spiral organism, *Leptospira icterohemorrhagiae*, which produces a severe and sometimes fatal jaundice in man, and which has a wide distribution in nature—far wider, indeed, than has yellow fever. This spiral microbe inhabits the rat, from which animal it finds its way at times to human beings, who then develop the symptoms and organic lesions of the disease called infectious jaundice.

I have given much attention to the spiral group of microörganisms for about ten years. In this period I have been enabled to devise new methods of cultivation, which have proven successful for species not before cultivated in successive generations, and which have also led to the discovery of new species. Hence in going to Guayaquil I took with me a laboratory outfit complete for this line of study.

Fortunately, I was successful in detecting in certain cases of yellow fever by culture methods and by guinea pig inoculation a particular spiral organism which I have since named *Leptospira icteroides*. The guinea pigs (and, as later studies showed, also puppies) successfully inoculated with the blood of yellow fever patients or with the cultures develop symptoms and lesions closely approximating those occurring in man and giving the clinical picture of the disease as usually observed. The outstanding signs are jaundice, hemorrhage into the lungs and stomach (this latter in man leads to the black vomit, so called), and albumin and casts in the urine. At autopsy, in the guinea pig as in man, the liver, kidneys, and other internal organs prove to be severely degenerated. The spiral organisms are, of course, recoverable from the inoculated guinea pigs, and with these organisms the disease is transmissible through an indefinite series of animals.

Moreover, guinea pigs have been successfully infected with the spiral organisms by means of Stegomyia mosquitoes, the vector in nature of the inciting microbe from man to man, and Stegomyias fed on infected guinea pigs are capable of transmitting the active microbe to still other guinea pigs, which develop the symptoms and lesions described.

Finally, immunological studies have brought out important points of relationship (for example, to the Leptospira of infectious jaundice) and indicated the possibility of developing a "vaccine," and even a curative serum. But until the finding of *Leptospira icteroides* is confirmed by the investigation of cases of yellow fever in still other places, its standing as the inciting agent of yellow fever will have to be regarded as not yet certainly established.

HUMAN RETINAL ADAPTATION

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Communicated by J. Loeb, January 2, 1920

Although the magnitude and the course of the dark adaptation of the human eye are well known, no fundamental interpretation has been made of the results. The data are simple. In the dark the eye becomes increasingly sensitive to light as the time goes on, until after about an hour it reaches a maximum of irritability. The measure of this changing sensitivity is given in terms of the least intensity which is just perceptible to the eye. The data which have been accumulated in this way are plentiful and accurate.[1]

The lack of theoretical significance of these data is caused by two conditions. The first is that the term sensitivity, even as measured in units of intensity, has no ultimate material basis. The fundamental question is: what objective changes in the makeup of the retina determine the variations in its sensitivity? This cannot at present be answered from experiments on the eye. However, even if this situation were satisfactorily elucidated, a second limitation bars the way to a final interpretation. This second condition hinges on the photochemical effect of the light. Since the data of retinal dark adaptation are given in terms of intensity, it is necessary to have some quantitative estimate of the photolytic action of light at the various intensities used for stimulation. This condition again cannot be fulfilled from the reported experiments with the eye. Indeed, in the present state of our knowledge it is difficult to devise experiments for the purpose of investigating these two conditions.

Fortunately the physicochemical analysis of the photoreception of lower forms has reached the point where these questions may be answered with considerable accuracy and a fair degree of confidence.[2] With regard to the first limitation of retinal interpretation, the experiments with *Mya* and *Ciona* are unequivocal. A photosensitive substance S is decomposed by the light into its two precursors P and A. This reaction is reversible, the regeneration of photosensitive material S from its two precursors P and A following the course of a bimolecular reaction. Thus the equation of the reaction is

$$S \underset{\text{"dark"}}{\overset{\text{light}}{\rightleftharpoons}} P + A.$$

The important point in this mechanism is that the sensitivity of the sense organ at any moment is dependent on the concentration of precursor substances P and A. Quantitatively expressed this means that the amount of fresh precursors that must be formed by the light before a

response is elicited is always a constant fraction of the amount of precursors already present in the sense organ. Conversely, this amount of fresh precursors is a direct measure of the concentration of residual precursors. If this concentration is high, much fresh precursors are required, and much light energy must be used to produce the minimum stimulating effect. If the concentration of residual precursors is low, the reverse will be true. In short, the sensitivity is thus defined in terms depending on the physical constitution of the sense organ.

Similarly with the second condition which inhibits the interpretation of retinal adaptation. Experiments to be published in the *Journal of General Physiology* show a simple relation between the intensity of the stimulating light and its photochemical effect in the photosensory process. If E is the photolytic effect as measured in units of P and A formed, then the results are that

$$E = \ln I$$

where I is the intensity of the stimulating light. The photochemical effect of the light is, therefore, a logarithmic function of its intensity. Here again the meaning of the minimum intensity is given in terms of the ultimate composition of the sensory mechanism.

If the initial phase of the photic reception of the human retina in dim light is similar to the mechanism in these lower forms, the application of the two principles just described should yield an analysis of the results obtained in dark adaptation. The sensitivity of the eye given in terms of the minimum intensity of the stimulating light can now be replaced by a direct statement of the photochemical effect of this light. Instead of the intensity we may use the logarithm of the intensity to indicate the extent of this photolytic activity. This now shows that during dark adaptation of the eye less and less photochemical decomposition is required to produce a visual effect. Assuming a photosensitive substance in the retina, this means that as the time goes on less and less of this photosensitive material must be decomposed in order to produce a stimulus. The exact amount of this decomposition can now be calculated.

If in addition it is true for the eye that the amount of freshly formed decomposition product is always a constant fraction of the residual decomposition product present in the retina, the significance of these facts becomes clear. During dark adaptation less and less fresh decomposition product must be formed. Therefore, the quantity of residual decomposition product present in the retina must also become less and less, and at the same rate as that of the fresh decomposition product. The ratio between the two being constant, the curve of dark adaptation should represent the rate of disappearance of the residual decomposition product.

The many experiments reported on the retinal adaptation of the eye, when analyzed in this way, show a uniformly consistent course for the dis-

appearance of the decomposition product. The curve which represents all of the published data is the isotherm of a bimolecular reaction

$$k = \frac{1}{at} \cdot \frac{x}{a - x}$$

in which k has an average value of approximately 0.016. Because of the refined technic used in the experiments the agreement between the individual observations and the theoretical curve is frequently diagrammatic in its precision. An example is given in figure 1. The observations are

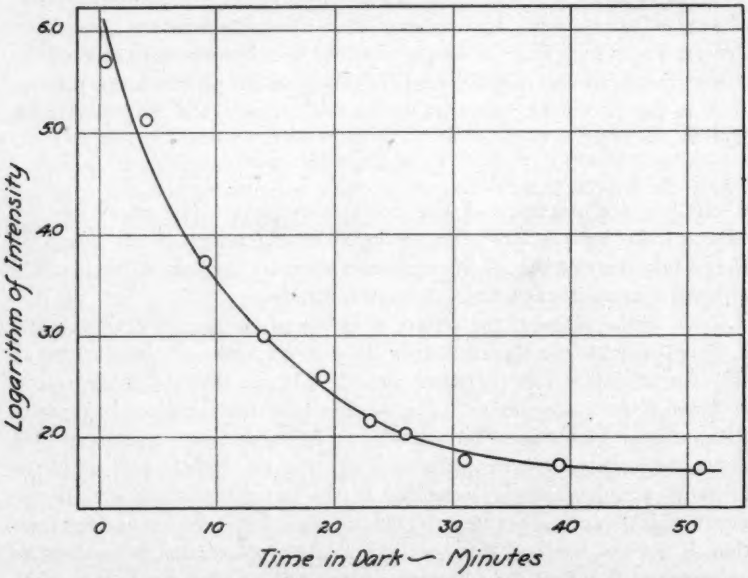

FIG. 1. Dark adaptation of the human eye. The points are single observations, whereas the curve is the isotherm of a bimolecular reaction. Actually the ordinates are the logarithms of the minimum intensities at different periods in the dark; theoretically they represent the number of units of decomposition products still present in the retina.

taken from an experiment of Nagel's made more than ten years ago.

The meaning of the representation of the course of retinal dark adaptation by the curve of a bimolecular reaction is obvious. *Two* substances are decreasing in concentration according to the ordinary velocities of chemical reactions. These two substances are, therefore, combining to form a third substance.

The simplest interpretation of these facts is that the substance synthesized during dark adaptation is the photosensitive substance, and that the two materials which form this photosensitive substance are its precursors as well as its decomposition products. The initial process in visual reception in dim light must, therefore, depend on a reversible photochemical

reaction in which a photosensitive substance is broken down into *two* decomposition products. During dark adaptation of the retina these two decomposition products then unite to form again the sensitive substance from which they were produced. The actual course of dark adaptation depends on the change in concentration of these two reacting substances.

In all essentials, then, the mechanism underlying the initial phase of retinal sensitivity in dim light is the same as that which forms the basis of the initial process of photoreception in such forms as *Mya* and *Ciona*.

[1] W. Nagel, in Helmholtz' *Handbuch der physiologischen Optik*, **2**, 1911.

[2] S. Hecht, *Journal of General Physiology*, **1** and **2**, 1918–20.

A KINEMATICAL INTERPRETATION OF ELECTROMAGNETISM

BY LEIGH PAGE

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The concept of lines of force was introduced by Faraday as an aid in mapping out magnetic fields—these lines having everywhere the direction in which a small north pole would be urged by the field. Later it occurred to him that the characteristics of an electric field might be represented in similar manner by curves everywhere tangent to the electric intensity. Faraday limited his use of lines of force to static fields, and in cases where the field was produced by a number of charged particles, to the resultant field.

The electromagnetic theory of light, however, made it necessary to suppose that electric and magnetic lines of force in a wave front move with the velocity of light in a direction at right angles to their plane. Furthermore, the discovery of the electron made it natural to assume that when one of these small particles is in motion, its field is carried along with it. Wiechert and Stokes have explained X-rays as kinks in the lines of force emanating from such a particle—these kinks being propagated outward with the velocity of light whenever there is a change in the electron's state of motion.

The object of this paper is to show that the laws of electromagnetism may be explained exactly and in their entirety as kinematical relations between the moving elements which constitute lines of force, provided use is made of the space and time transformations of the principle of relativity in place of the more familiar but less philosophical Galilean transformations.[1] Each charged particle must be considered to have its own field, extending out to infinity in all directions, the resultant field at any point being a combination of those fields which extend to the point in question.

The electrodynamic equations are five in number: four of these specify the electric and magnetic fields in terms of the distribution of charge and current throughout space, and the fifth specifies the force on charged matter due to the strength of the electric and magnetic fields in which it is moving. First will be taken up the kinematical interpretation of the two equations —Coulomb's law and Ampere's law—which do not involve the space time transformations, and hence are not dependent on the principle of relativity.

In the representation to be employed, each charged particle e is to be considered as a source of moving elements projected outward uniformly in all directions with the velocity of light c. A line of force is a curve drawn through the moving elements which have been projected in a given direction. Perhaps a better picture of the method by which the field is produced may be obtained by imagining a very large number of guns to be stationed at e, pointing outward in all directions. Suppose each gun to be provided with a long, perfectly elastic cord on which shot are strung. Let these shot be fired from the gun at regular but very short intervals with the velocity of light. Then the shot constitute the moving elements emanating from the charge, and the elastic cord connecting them marks the line of force diverging in the particular direction under consideration.

The number of lines of force diverging from a charge de will be supposed to be very large, no matter how small de may be. For convenience in making numerical calculations, these lines will be grouped into bundles or tubes of M lines each, where M is a large number so chosen as to make the number of tubes N diverging from any charge equal numerically to the magnitude of the charge. This convention is tantamount to measuring electric charges in Heaviside-Lorentz rational units.

If the electric intensity $\mathbf{E}$ is defined as the number dN of tubes per unit cross section, the component of the intensity normal to the small surface $d\sigma$ is

$$\mathbf{E}_{d\sigma} \equiv \frac{dN}{d\sigma^2}\, d\sigma.$$

According to Gauss' theorem, the average divergence of $\mathbf{E}$ over a small volume $d\tau$ surrounded by the closed surface σ is given by

$$\nabla\cdot\mathbf{E}d\tau = \int_\sigma \mathbf{E}\cdot d\sigma$$
$$= \int_\sigma dN.$$

Now the part of this sum due to charges outside the region $d\tau$ vanishes, while the part due to the charge de inside this region is equal to de itself. Hence, if ρ is the density of charge at the point in question,

$$\nabla\cdot\mathbf{E} = \rho. \qquad (1)$$

This is Coulomb's law.

To obtain Ampere's law, consider the increase in electric intensity at O (fig. 1) taking place in the time dt. During this time the lines of force

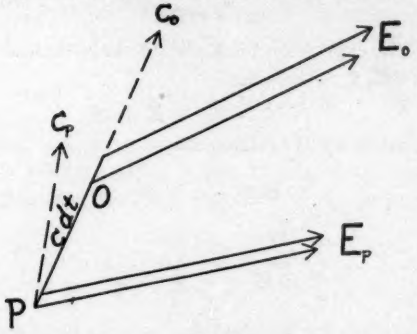

FIG. 1

at P will have moved up to O. If these lines are closer together at P than at O, $\mathbf{E}$ will change by an amount

$$-\mathbf{c}\cdot\nabla\mathbf{E}dt.$$

Furthermore $\mathbf{E}$ will suffer a change if the velocity $\mathbf{c}_P$ of the moving elements at P has a different direction from that at O, or if these lines twist as they move upward. The increase in $\mathbf{E}$ due to these two causes is

$$(-\mathbf{E}\nabla\cdot\mathbf{c} + \mathbf{E}\cdot\nabla\mathbf{c})dt.$$

Therefore, in all,

$$\dot{\mathbf{E}} = -\mathbf{c}\cdot\nabla\mathbf{E} - \mathbf{E}\nabla\cdot\mathbf{c} + \mathbf{E}\cdot\nabla\mathbf{c}$$
$$= -\mathbf{c}\nabla\cdot\mathbf{E} + \nabla\times(\mathbf{c}\times\mathbf{E}).$$

So far, no account has been taken of the new lines of force produced by the emission of moving elements in the neighborhood of a charged particle.

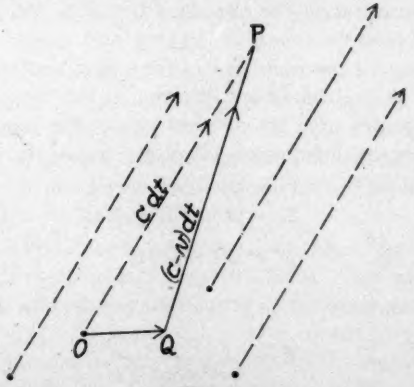

FIG. 2

Let the dots in figure 2 represent sources emitting lines of force in the direction of the broken lines, and let these sources have a velocity $\mathbf{v}$ to the

right. Then the rate of change of the electric intensity at P due to the formation of new lines is

$$(\mathbf{c} - \mathbf{v})\nabla\cdot\mathbf{E}.$$

Adding this to the expression for $\dot{\mathbf{E}}$ obtained above, and using Coulomb's law to eliminate $\nabla\cdot\mathbf{E}$, it is seen that

$$\nabla \times (\mathbf{c} \times \mathbf{E}) = \dot{\mathbf{E}} + \rho\mathbf{v}.$$

The magnetic intensity $\mathbf{H}$ is defined by

$$\mathbf{H} \equiv \frac{1}{c}\,\mathbf{c} \times \mathbf{E}.$$

Therefore

$$\nabla \times \mathbf{H} = \frac{1}{c}(\dot{\mathbf{E}} + \rho\mathbf{v}). \qquad (2)$$

This is Ampere's law for the field due to a single charged particle. As it is linear in $\mathbf{E}$ and $\mathbf{H}$, the same law holds for the resultant of the simple fields due to any number of charged particles.

Next consider a charged particle e (fig. 3) at rest relative to the observer

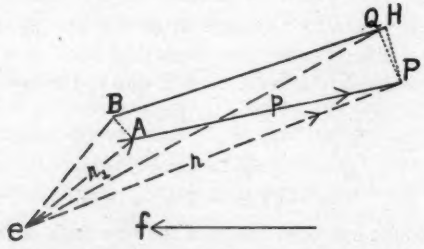

FIG. 3

but having an acceleration f to the left. Let eP be the path of a moving element emitted from the charge at the time zero, and eA that of a moving element belonging to the same line of force, but emitted at the time dt. As the velocity of the charge has changed in this time, these paths will make different angles with the acceleration. The line of force at P at time r/c will have the direction of AP, or $\mathbf{p}$. Evidently

$$\mathbf{r} = \mathbf{c}t,$$
$$\mathbf{r}_1 = \mathbf{c}t - \mathbf{c}dt + td\mathbf{c},$$
$$\mathbf{p} = \mathbf{c}dt - \frac{r}{c}\,d\mathbf{c},$$

and the electric intensity at P at the time r/c is given by

$$\mathbf{E} = \frac{e}{4\pi r^2}\,\frac{\mathbf{p}}{p\cos QPH}$$
$$= \frac{e}{4\pi r^2 c}\left\{\mathbf{c} - \frac{r}{c}\,\frac{d\mathbf{c}}{dt}\right\}.$$

A short calculation gives

$$\frac{d\mathbf{c}}{dt} = -\frac{1}{c^2}(\mathbf{f} \times \mathbf{c}) \times \mathbf{c}.$$

Therefore,

$$\mathbf{E} = \frac{e}{4\pi r^2 c}\left\{\mathbf{c} + \frac{r}{c^3}(\mathbf{f} \times \mathbf{c}) \times \mathbf{c}\right\}. \quad (3)$$

So far no use has been made of the relativity transformations. In order to get the electric intensity due to a moving charge, recourse must be had to these transformations, resulting in the following expression,

$$\mathbf{E} = \frac{e}{4\pi r^2 k^2 c\left(1 - \frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)^3}\left\{\mathbf{c} - \mathbf{v} + \frac{rk^2}{c^3}\{\mathbf{f} \times (\mathbf{c} - \mathbf{v})\} \times \mathbf{c}\right\}, \quad (4)$$

where

$$k \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

From the definition of magnetic intensity, it follows that

$$\mathbf{H} = \frac{e}{4\pi r^2 k^2 c\left(1 - \frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)^3}\left\{-\frac{\mathbf{c} \times \mathbf{v}}{c} + \frac{rk^2}{c^4}\mathbf{c} \times (\{\mathbf{f} \times (\mathbf{c} - \mathbf{v})\} \times \mathbf{c})\right\}. \quad (5)$$

For fields specified by these expressions, it is easily shown that

$$\nabla \mathbf{H} = 0, \quad (6)$$

and

$$\nabla \times \mathbf{E} = -\frac{1}{c}\dot{\mathbf{H}}. \quad (7)$$

These are the two remaining field equations, the second being the mathematical expression of Faraday's law of current induction for the case where a current is induced by varying the magnetic flux through a stationary circuit. As these laws are linear in **E** and **H**, they hold as well for the resultant of the simple fields due to a number of charged particles as for each of these fields individually.

The last of the electromagnetic equations gives the force on a charge moving through electric and magnetic fields. Let unprimed letters in figure 4 refer to quantities as measured in a so-called stationary system, and primed letters to the same quantities as measured in a system moving to the right with velocity **v**. On account of the dependence of time on position required by the relativity transformations, the line of force PE in the stationary system has the different direction PE' in the moving

system. If the dynamical equation for a charge e situated in the moving system is assumed to be

$$e\mathbf{E}' = \frac{d}{dt'}(m\mathbf{v}'),$$

as measured in the units of this system, the relativity transformations give

$$e\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{H}\right) = \frac{d}{dt}\left(\frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}\mathbf{v}\right) \tag{8}$$

in the stationary system. This is the fifth and last of the electromagnetic equations. If the relativity transformations had not been made use of in deriving it, the term involving the magnetic intensity would have been absent. But this term is the one that accounts for the current induced

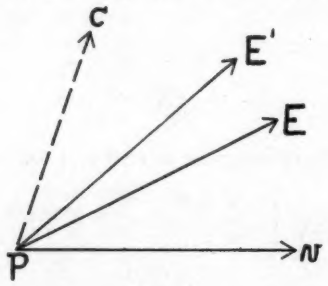

FIG. 4

in a wire moving through a steady magnetic field, and for the torque in a coil carrying a current which is placed in such a field. Therefore, the action of every generator and every motor used by industry in this age of electricity is incontrovertible evidence of the truth of the principle of relativity.

In deriving expression (3) for the electric intensity due to a point charge, it has been tacitly assumed that the field does not rotate. If the sources of moving elements constituting the charge—the guns in the analogy previously used—are supposed to rotate with angular velocity ω, the rate of change of $\mathbf{c}$ at the time r/c becomes

$$\frac{d\mathbf{c}}{dt} = -\frac{1}{c^2}[(\mathbf{f} \times \mathbf{c}) \times \mathbf{c} + \{(\omega \times \mathbf{c}) \times \mathbf{c}\} \times \mathbf{c}],$$

for a charge which is at rest relative to the observer at the time zero. This introduces into the expression (3) for $\mathbf{E}$ the additional term

$$\frac{e}{4\pi r^2 c}\left\{\frac{r}{c^3}\{(\omega \times \mathbf{c}) \times \mathbf{c}\} \times \mathbf{c}\right\},$$

and $\mathbf{E}$ and $\mathbf{H}$ for a moving charge become

$$\mathbf{E} = \frac{e}{4\pi r^2 k^2 c\left(1 - \frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)^3}\left\{\mathbf{c} - \mathbf{v} + \frac{rk^2}{c^3}\{\mathbf{f} \times (\mathbf{c} - \mathbf{v})\} \times \mathbf{c} + \frac{r}{c^3}(\{\omega \times (\mathbf{c} - \mathbf{v})\} \times \mathbf{c}) \times \mathbf{c}\right\}, \quad (9)$$

$$\mathbf{H} = \frac{e}{4\pi r^2 k^2 c\left(1 - \frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)^3}\left\{-\frac{\mathbf{c} \times \mathbf{v}}{c} + \frac{rk^2}{c^4}\mathbf{c} \times (\{\mathbf{f} \times (\mathbf{c} - \mathbf{v})\} \times \mathbf{c}) + \frac{r}{c^2}\{\omega \times (\mathbf{c} - \mathbf{v})\} \times \mathbf{c}\right\}. \quad (10)$$

The field equations (1) and (2) representing Coulomb's law and Ampere's law are unchanged for this more general type of field, but (5) and (6) become

$$\nabla\cdot\mathbf{H} = \epsilon, \quad (11)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}(\dot{\mathbf{H}} + \epsilon\mathbf{u}), \quad (12)$$

where the density of magnetic charge ϵ is

$$\epsilon \equiv \frac{e}{2\pi r^2 k^2 c^2} \frac{\frac{\omega\cdot\mathbf{c}}{k^2} - \omega\cdot\mathbf{v}\left(1 - \frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)}{\left(1 - \frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)^4}, \quad (13)$$

and its velocity u is in general equal to the velocity c of light. The energy equation for fields of this character takes the form

$$\frac{d}{dt}\left\{\frac{1}{2}(E^2 + H^2)\right\} + c\nabla\cdot(\mathbf{E} \times \mathbf{H}) + \mathbf{E}\cdot\mathbf{v}\rho + \mathbf{H}\cdot\mathbf{u}\epsilon = 0, \quad (14)$$

and the energy radiated by a charge e due to rotation of the field is

$$R = \frac{e^2\omega}{3c}$$

for each revolution. In terms of the frequency ν this expression has the form

$$R = \frac{2\pi e^2}{3c}\nu.$$

If the value of the electronic charge is substituted for e, to wit

$$e = 4.77\ (10)^{-10}\sqrt{4\pi}$$

Heaviside-Lorentz units, it is found that

$$R = \frac{1}{32}\ 6.4(10)^{-27}\ \nu \text{ erg sec.}$$

$$= \frac{1}{32}\ h\nu,$$

where h is Planck's constant. This numerical coincidence suggests that quantum phenomena may be in some way connected with rotating electric fields, and that

$$h \equiv \frac{64\pi e^2}{3c}.$$

Summary.—The equations of electrodynamics are shown to be simple kinematical relations between the moving elements which constitute lines of force. If the sources of these moving elements are supposed to rotate, free magnetic charges are produced, and the four equations of the electromagnetic field assume a symmetrical form. The energy radiated due to the rotation is found to amount almost exactly to $^1/_{82}\ h\nu$ for each revolution.

[1] The reader who is interested in following through in more detail the analysis involved in deriving these relations is referred to the author's paper on "Relativity and the Ether," *Amer. J. Sci., New Haven*, **38**, 1914 (169).

THE LAWS OF ELASTICO-VISCOUS FLOW. II

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Communicated January 19, 1920

In a paper of Harold Jeffreys entitled "The Viscosity of the Earth,"[1] the author makes use of a formula which combines the laws of Larmor and of Maxwell.

$$n\left(S + \tau_2 \frac{ds}{dt}\right) = F + \frac{1}{\tau_1}\int F dt.$$

The integral implies a permanent set which as the author indicates would be inconsistent with the "accepted theories of tidal friction and variation of latitude. Hence τ_1 must be practically infinite." The formula is thus reduced to the expression $F = n_1(S + \tau_2\, ds/dt)$.

Experiments made on a great variety of materials show, however, that this expression must be seriously modified to represent the facts.

Thus it has been shown[2] that the displacement produced by a stress P is given by the expression[3] $S = C_1Pe^{h_1P} + C_2Pe^{h_2P}(1 - e^{-\alpha\sqrt{i}}) + C_3Pe^{h_3t^\rho}$. The last term produces permanent set, so for the present may be omitted. Putting

$$C_1 = \frac{1}{n_1} \text{ and } C_2 = \frac{1}{n_2}\ \alpha = \frac{1}{\tau}\ \theta = t^{1/2}$$

this becomes

$$S = \frac{P}{n_1}\left(A_1 + \frac{n_1A_2}{n_2}\left(1 - e^{-\frac{\theta}{\tau}}\right)\right)$$

whence

$$n_1\left(S - \tau\frac{ds}{d\theta}\right) = P\left(A_1 + \frac{n_1}{n_2}A_2\right).$$

For small stresses $A_1 = A_2 = 1$ and if n_2 equal n_1 this expression takes a form resembling that given by Jeffreys.

It is important to note, however, that this formula is based on the assumption that the viscosity is "external"—that is, it acts as though the viscous resistance were due to an absolute velocity ds/dt. But this is by no means evident; and indeed the probability is that a considerable part if not the major part of the viscous resistance may be "internal" —that is, due to the relative motion of parts. Thus if an element consist of two parts y and z, y being coupled to the next adjacent element by an elastic coupling n_1 and with z by an elastic coupling n_2 together with a viscous coupling ϵ_2 while ϵ_1' and ϵ_2' represent the "external" viscosities, the equations of motion will be

$$\rho_1 \ddot{z} = \epsilon_2(\dot{z} - \dot{y}) + n_2(z - y) + \epsilon_1'\dot{z}$$
$$\rho_2 \ddot{y} = \epsilon_2(\dot{z} - \dot{y}) + n_2(z - y) + \epsilon_2'\dot{y} + n_1 \frac{d^2y}{dx^2}.$$

If ρ_2, ϵ_1' and ϵ_2' be considered negligible, the solution, for not too rapid extinction, is

$$z = ae^{-\beta x}\cos p(t - vx)$$

in which

$$v^2 = \frac{n_1}{\rho} \frac{(n_2 - \rho p^2)^2 + p^2\epsilon^2}{n_2(n_2 - \rho p^2) + p^2\epsilon^2}$$
$$\beta = \frac{\rho^{1/2} p^4 \epsilon}{2\sqrt{n}[n_2(n_2 - \rho p^2) + p^2\epsilon^2]}.$$

If pe is large compared with n_2

$$v^2 = \frac{n}{\rho}\left(1 + \frac{\rho^2 p^2}{\epsilon^2}\right)$$
$$\beta = \frac{\rho p^2}{2\epsilon v}.$$

So that in this case the higher the viscosity the less rapid the decay of the oscillations—quite the reverse of the conclusions on the former assumption. But the appearance of $\Theta = t^{1/2}$ is a more serious matter, making the use of the formula much more difficult.

The operator which should replace n is, therefore,

$$n_1 \frac{1 + 2\tau\sqrt{t}\,\frac{ds}{dt}}{1 + \frac{n_1}{n_2} A_2}.$$

But the application of this formula to such a problem as the earth's viscosity is still further complicated by the fact that all the constants are functions of the pressure and of the temperature in the earth's interior. Even though more or less probable assumptions may be made regarding

the value of temperature and pressure as functions of the distance from the center, we know but little regarding the effect of these factors on either rigidity or viscosity.

1. It was found that the temperature effect may be represented with considerable accuracy by the expression

$$A = Ee^{(K + bP)\Theta}$$

in which P is the applied stress, Θ the temperature and E, K and b constants.

For room temperature the values of $b\Theta = h$ are given in table IV.

If we take $h = 0.2$ as fairly representative

$$S_2 = Pe^{.2P}(1 - e^{-\sqrt{t}}). \quad \text{(The unit } P = 100 \text{ gm.)}$$

so that G the couple $= Pr$ gm. cm. Thus we get for the displacement after a sufficiently long time

$$S = Pe^{.2P} \text{ and } \frac{1}{n_2} = \frac{S}{P}$$

TABLE I

P	S/P
0	1.0
1	1.2
10	7.0
50	20000.0

showing the very great increase in importance of the elastico-viscous term for large stresses. The same is also true for the purely viscous term.

The following table gives the ratio n_2/n_1 for twenty-two materials showing that there are certainly two elasticities—one of which is not accompanied by viscosity and the second is thus affected. In every case excepting that of sealing wax, where the ratio is unity, the second elasticity is much greater than the first, and in some cases enormously greater.

TABLE II

	$\frac{n_2}{n_1}$		$\frac{n_2}{n_1}$
Tin	60	Slate	150
Zinc	70	Shale	50
Marble	45	Soapstone	35
Limestone	30	Lead	40
Ebonite	8	Cadmium	65
Iron	2200	Gold	150
Steel	12000	Magnesium	250
Copper	8	Bapelite	7
Aluminum	4400	Ivory	50
Talc	200	Silver	80
Glass	100	Sealing Wax	1

The introduction of $t^{1/3}$ instead of t itself is a step so radical that it may be well to give an illustration in its justification.

For this purpose it is desirable to choose a material in which the elastico-viscous effect is well marked. This is notably the case for vulcanite, which has the added advantage of the relatively small importance of the third or purely viscous term. This illustration is, perhaps, the most striking in showing the appropriateness of $t^{1/3}$ instead of t; but all the materials investigated give similar results.

Following is a table of results for R_o, the return at the time t after releasing the stress.[4] ${}^{R}\sqrt{t}$ gives the result of calculation from

$$R = 890(1 - e^{-.57\sqrt{t}})$$

R_t gives values calculated from

$$R = 840(1 - e^{-.4t}).$$

The differences between calculated and observed values under Δ_1 and Δ_2 show that the former expression is very near the truth, while the latter is entirely inadequate.

TABLE III

t	R_o	R_t	Δ_1	R_t	Δ_2
1	380	387	7	277	—103
2	490	492	2	462	—28
4	600	605	5	672	+72
9	730	729	—1	820	+90
16	800	802	2	838	+38
25	840	841	1	840	00
30	853	851	—2	840	—13
00	890	890	0	840	—50

While the term involving a permanent set may not have any application to the problem of the Earth tides, yet it may not be amiss to draw attention to the fact that in some cases and especially at temperatures approaching the melting-point this term becomes the most important of all. The temperature coefficient in this case enters in the form $\Theta/T - \Theta$; giving as it should perfect fluidity at T, the melting-point.

In the former article the expression given for this viscous term is $S_3 = (Ft)^\rho$ in which $F = C_3 P e^{h_3 P}$ and ρ is stated to be approximately one-half.

From more recent data the average value of ρ is 0.41; and if from the nineteen substances examined, four be excluded the average is 0.35 which makes it much nearer one-third than one-half.

The expression for the viscous terms should be $S_3 = (Ft)^{1/3}$ if the stress (P) is constant. If P is a function of time

$$S_3 = (\int F dt)^{1/3}.$$

Thus if P be given a constant value for a time t^0 and then changed to P_1 the corresponding value of the viscous term would be

$$S_3 = (F_0 t_0 + F_1 t)^{1/3}.$$

If the first stress be considerable and act for a long time, the effect of the second stress is negligible.

Following is a provisional table of the constants which appear in the formula for the torsional strain at room temperature.

$S = A_1 + A_2 (1 - e^{-\alpha\sqrt{t}}) + A_3 t^\rho + A_4$ in which $A = CPe^{hP}$, P is the weight acting on a pulley of radius 5 cm. The unit of weight being 100 grams and the unit of time one minute. The specimen is a cylindrical rod 7.5 cm. long and 4 mm. in diameter.

The term A_4, which may be termed the "lost motion" should probably be considered as a part of the viscous term—but with a very small exponent r so that the whole viscous term may be represented by

$$S_3 = C_3 P e^{h_3 P}(t^\rho + B t^r).$$

TABLE IV

	C_1	C_2	C_3	C_4	h_1	h_2	h_3	h_4	α	ρ
Tin slowly cooled.....	660	20.0	3.0	1.0	0.10	1.0	3.5	4.5	0.8	0.6
Tin quickly cooled.....	640	8.0	0.6	0.3	0.00	1.8	3.8	4.6	0.8	0.6
Zinc slowly cooled.....	312	3.0	3.4	2.0	0.00	0.4	0.3	0.8	1.0	0.5
Zinc quickly cooled.....	300	8.4	3.0	0.00	0.00	0.4	0.6	1.8	1.0	0.5
Marble.....	840	18.0	40.0	34.0	0.14	1.8	0.8	0.3	1.1	0.2
Limestone.....	600	20.0	9.0	11.0	0.10	0.3	0.3	0.8	1.2	0.3
Ebonite, first determ....	13 x 10^3	1600	600?	0?	0.00	0.1	0.0		0.6	0.5
Ebonite, second determ..	13 x 10^3	1800.0	750.0	0.0	0.00	0.1	0.2		0.9	0.7
Soft Iron Annealed.....	145	0.06	0.00	0.15	0.00	0.1		0.1	1.0	0.2
Soft Iron unannealed....	155	0.08	0.00	0.15	0.04	0.1		0.1	1.0	0.2
Tool steel, annealed.....	144	0.01	0.00	0.10	0.00	0.2		0.1	0.4	0.3
Tool steel, glass hard....	143	0.14	0.26	0.07	0.00	0.0	.03	0.06	0.4	0.4
Copper, annealed.....	250	1.3	0.00	0.75	0.06	0.1	1.8	1.2	0.5	0.3
Aluminum, annealed....	440	0.1	0.00	0.9	0.00	1.0	2.5	1.4	1.0	0.3
Talc, parallel cleavage ..	1340	4.0	20.0	110.00	0.60	3.0	1.4	1.2	1.0	0.3
Slate, parallel cleavage..	382	2.0	0.4	0.6	0.01	0.3	0.7	0.6	1.2	0.3
Slate, perp. to cleavage..	400	4.0	1.6	1.0	0.03	0.3	0.00	0.8	1.5	0.6
Sealing wax.....	3 x 10^4	2 x 10^4	?	?	0.00	0.0	00.00	.00	0.7	?
Glass (plate).....	446	3.0	0.00	4.0	0.00	0.0	00.00	.00	0.5	
Glass (lead).....	425	5.6	0.00	0.1	0.00	0.0	00.00	.15	1.4	
Shale, parallel cleavage..	400	8.4	1.0	1.2	0.02	0.16	00.35	.6	1.2	0.4
Lead.....	1500	40.0	7.0	4.0	0.00	0.2	1.9	3.6	1.1	0.4
Cadmium.....	460	70.0	5.4	2.4	0.00	0.2	0.6	0.8	1.0	0.5
Gold.....	450	3.0	0.2	1.0	0.00	0.4	0.2	0.3	1.0	0.3
Magnesium.....	2300	9.4	8.0	50.0	0.00	0.25	0.4	0.6	1.2	0.3
Bakellite.....	7400	960.0	14.0	0.00	0.00	0.12	0.2		1.2	0.8
Ivory.....	10^4	320.0	3.0	0.00	0.00	0.10	0.2		0.8	0.6
Silver.....	480	6.0	2.0	0.4	0.00	0.06	0.1	0.2	1.0	0.3

[1] *Mon. Not. Roy. Astron. Soc.*, **77**, No. 5.

[2] These PROCEEDINGS, **3**, No. 5.

[3] The strains in these experiments were torsional, thus involving only the rigidity

constant n. In the formula as given in the paper referred to the coefficients C and the exponents h are functions of the temperature. The stress P is constant and ρ is approximately one-half.

[4] It was found by experiment that for stresses not too great, the "direct" curve (on applying the stress) and the "return" curve (on releasing) were the same; or rather if the former is S and the latter R, then $S - R = Ct$.

NOTE ON THE SIMPLE DEVICE FOR INCREASING A PHOTOGRAPHIC POWER OF LARGE TELESCOPES

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Communicated by G. E. Hale, February 5, 1920

Among the many problems in sidereal astronomy that demand great telescopic power the three following may be cited as particularly significant for inquiries relating to the origin and evolution of stellar and galactic systems: (1) The total number of stars and limiting faintest magnitude in globular clusters; (2) the frequency and distribution of dwarf stars in clusters and in the sky at large; (3) the limits of the galactic system in certain directions. To these stellar problems we should add the highly interesting questions connected with the faint extra-galactic nebulae, and note that important contributions toward their solution seem to be only a little beyond our present telescopic power. Thus the best available photographs of globular clusters suggest that we are approaching the faintest magnitudes, and that, if we could only extend our studies two or three magnitudes farther, one phase of the problem of dwarf stars could probably be solved.

The longest exposures with the 60-inch reflector have yielded apparent photographic magnitudes between 20 and 21. The Hooker telescope, it is believed, will gain about a magnitude over the 60-inch provided the focal images are not much larger; but in the case of nebulae, since the ratio of focal length to aperture is the same for the two reflectors, no gain is to be expected except in scale, which is of course important for revealing new structure and purposes of measurement, but will not bring fainter structure to view. Small nuclei will also be shown in stronger contrast with the 100 inch telescope.

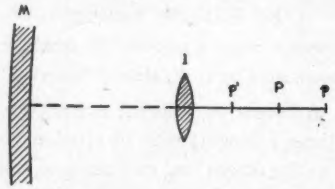
The investigation of faint stars and nebulae is of sufficient importance to justify every attempt to increase beyond normal limits the working range of great reflectors. Since these instruments are universally adapted to photographic observation, the problem is one of increasing the brightness of the image or increasing the sensitivity of the photographic plate.

In the device described below the principle employed is to increase the brightness of the image by reducing its area. This is accomplished by placing a short focus lens in the converging beam at an appropriate distance in front of the photographic plate. The immediate result is to re-

duce the effective focal length of the telescope, thus giving high speed and reducing the scale on the photograph.

In studies where the important thing is scale, an intensifying lens does not help; further, if a large field is desired, the intensifiers now available are unsatisfactory. But for the problems mentioned above, where scale and field are of second importance to limiting magnitude, the intensifying device is efficiently and inexpensively adaptable.

For example, by using such a device at the primary focus of the 100-



inch, so as to reduce the scale to that of the 60-inch, the exposure time needed to give a required density in the photograph of a nebula should be less than one-half that otherwise necessary, even after allowing for a reasonable loss of light in the lens system of the intensifier.

Several lenses have been experimented with successfully in combination with various telescopes during the last six months. I am indebted to Mr. Benioff, who was associated with me in the early plans and experiments, for assistance in adapting the different intensifiers.

Let

F_1 be the focal length of the objective (mirror) of linear aperature A, $\overline{MP}$

f be the focal length of the intensifier of linear aperture a, $\overline{Ip}$

f' be the apparent focal length of the intensifier in the converging beam, $\overline{Ip'}$

F be the equivalent focal length of the combination,

d be the distance between the objective and intensifier, $\overline{MI}$

$D = F_1 - d$ be the distance from the focus of the objective to the intensifier, $\overline{IP}$

$R = F/F_1$ be the reduction of focal length and linear scale of the field.

From the relation giving the focal length of a combination of lenses we derive,

$$F = \frac{F_1 f}{F_1 + f - d} = \frac{F_1 f}{f + D}, \qquad (1)$$

$$R = \frac{f}{f + D}. \qquad (2)$$

Equation (2) shows that for a given intensifier the reduction depends only on the distance of the intensifying element from the normal focal point, P, of the telescope. Since the reduction is independent of the apertures and the focal length of the objective (mirror), being a function only

of the focal length of the intensifier and its position in the cone of light, a given intensifier will reduce the focal lengths of all telescopes in the same ratio when placed at a fixed distance in front of the normal focal plane. The one now in use at Mount Wilson is suitable without structural alteration for work at both the primary and secondary foci of the 60-inch and 100-inch reflectors, and with the 10-inch photographic refractor of 45 inches focus. Conditions analogous to those above hold for amplifying lenses.

An interesting corollary of equation (2) is that no advantage accrues from having an intensifier of large aperture, unless, for the sake of accommodating a larger field, the increase in size can be made without increasing its focal length. Difficulties would obviously arise if the focal length of the intensifier were so short or so long that the distance of this secondary lens from the normal focus interfered with the photographic operations. Cf. equation (4) below.

The distances $\overline{IP} = D$ and $\overline{p'P} = D - f'$ are important in the actual manipulation of an intensifier. In the formula for conjugate foci,

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v},$$

we may put $u = f'$, $v = -D$, so that

$$\frac{1}{f'} = \frac{1}{f} + \frac{1}{D}, \tag{3}$$

and from (2)

$$f' = RD, \quad D - f' = D(1 - R). \tag{4}$$

That is, for any desired reduction of the effective focal length, the distance of the modified focus from the ordinary focal plane of the telescope is $1 - R$ times the distance of the intensifier from that plane. Equations (2) and (4) define the arrangement of a given intensifier and the photographic plate.

As an illustration, let us suppose that in working with the 100-inch Hooker telescope the reduction desired is $R = 1/3$, and that the focal length of the intensifier employed is $f = 3$ inches; then $D = 6$ inches, $D - f' = 4$ inches. The primary focus of the Hooker reflector is $F_1 = 42.3$ feet. From (1) the equivalent focal length is 14.1 feet, and we have, in effect, a 100-inch reflector with a focal ratio of 1.7 instead of 5. If we assume that not more than 25% of the light is lost in the intensifier, the theoretical gain of the telescope (limiting faintness) is a little more than two magnitudes.

For $R = 1/2$ the same intensifier should give a gain over normal photographic power of approximately 1.25 magnitudes, and leaves the scale very slightly inferior to that of the 60-inch reflector.

The most efficient intensifier for use with telescopes built for usual photographic purposes will be one that is adaptable to a large range of values of R; it should probably have a focal length between two and five

inches and the largest aperture possible, and should not be wasteful of light through reflections from numerous glass-air surfaces. If an extended object, rather than a point source, is to be photographed, the flatness of the intensifier field is important; or, better still, its focal properties should be such as to improve the field of the primary.

Putting $S = F/A$ and $s = f/a$ for the ratios of focal length to aperture for the objective (mirror) and intensifier, respectively, we find the minimum equivalent focal length, which is obtained when the cone of light from a source on the axis exactly fills the intensifier, is

$$F_{\min} = \frac{F_1}{1 + S/s};$$

also

$$D_{\max} = Sa.$$

For the Mount Wilson reflectors $S = 5$, and the greatest reduction of scale, for full accommodation of the convergent light, is

$$R_{\min} = \frac{1}{1 + 5/s}.$$

An intensifying lens with focal ratio 1.0 would, at the limit, increase the theoretical photographic limit by nearly four magnitudes for a source on the axis.

Intensifying devices are particularly suited to reflecting telescopes because of the freedom from chromatic aberrations; also they should be of especial value when used with instruments of largest aperture and longest focal length because in studies of faint nebulae the large scale allows the minification necessary for increased speed. Influences of bad seeing, or any other defects in the photographic images, are obviously minimized with the intensifier—a fact that may make it possible to use large reflectors under conditions otherwise impracticable.

The lens now in use is a Dallmeyer No. 2 Kinematograph of 3 inches focal length and focal ratio 1.9. It has a fine field, but the many glass-air reflections considerably diminish the light. The lens is mounted so that the reduction can be readily changed, with a range of R between $^5/_7$ and $^2/_7$ at the primary focus of the 100-inch reflector. For $R = {}^1/_2$ the workable field is about 6′ in diameter—quite sufficient for most clusters and nebulae; for $R = {}^1/_3$ it is one third as large. A comparative discussion of results obtained will be made in a later communication. The preliminary observations, which include successive exposures on the same plate with and without the intensifier, show that, when a reasonable allowance is made for loss of light in the lens, the predicted results are fully obtained, at least for nebulae. For instance, a three minute photograph of the spiral Messier 77, with a reduction of $^3/_8$, shows much fainter nebulosity than a ten minute exposure without the intensifier; and in the star cluster Messier 3 a reduction of $^8/_{15}$ gives a gain of nearly a magnitude for exposures of equal length.

ANTHROPOMETRY OF THE SIOUAN TRIBES

BY LOUIS R. SULLIVAN

AMERICAN MUSEUM OF NATURAL HISTORY, NEW YORK

Communicated by H. F. Osborn, January 14, 1920

The material is the result of investigations made on the occasion of the World's Columbian Exposition in Chicago in 1893 under the supervision of Professor Franz Boas. The measurements and observations were made by Franz Boas, F. C. Smith, J. W. Cooke, G. A. Kaven, Z. T. Daniels, E. F. Wilson, C. A. Helvin, F. C. Kenyon and G. M. West. The series contains male and female children and adults and includes a large series of half-bloods as well as full-bloods. The total series deals with 1431 individuals which makes it particularly valuable from a statistical point of view. Of this 1431, 594 were male adult full-bloods, 77 were male adult half-bloods, 181 were female adult full-bloods and 19 were female adult half-bloods. The remainder of the series are children.

Individuals from the following bands or tribes representing sub-divisions of the closely allied Siouan Indians were included in the series: Santee, Wahpeton, Sisseton, Yankton, Yanktonai, Cut Head, Teton, Brulé, Loafer, Oglala, Waziahziah, Sans Arc, Blackfoot Sioux, Minneconjou, Two Kettle, Hunkpapa and Assiniboine. In the larger adult male series each of these bands was represented on the average by about 30 men. The material was first studied by local bands, but so close an agreement was found in all the principal measurements and characteristics that it seemed preferable to include them all in one large series. The result is a fairly homogeneous group with a variability slightly below civilized groups in most characteristics.

Fourteen measurements were taken from which were calculated eight indices. Detailed observations were also made on the hair, skin, eyes, nose, ears, etc.

The series is of interest from two points of view: first, from the point of view of accurately describing and defining the Siouan type and showing its relationship to those American Indian tribes already described, and secondly from the point of view of racial intermixture since we have in this series individuals representing the results of the intermixture of two widely separated races of mankind.

On the first point, we may say that the Sioux with an average male stature of 1724 mm., are among the very tallest of the American Indians who range in stature from the 1530 mm. of the Guaranis of South America to the 1750 mm. of the Maricopa and Cheyenne. In head form they are mesocephalic with an index of 79.6. The face is very wide and high, 149.1 and 124.6 mm., respectively. On the average the face is nearly as wide as the head. The average proportion is 96.1%. The nasal bridge is rather high. The hair, of course, is straight and black, the eyes dark

brown, and the skin shows varying shades of brown. The beard, moustache and body hair is poorly developed.

Comparative material for estimating the affinities of the Siouan tribes is not abundant. In nearly every measurement and index the averages for our series are in very close agreement with the Chippewa (Ojibway) series of Hrdlicka. Less detailed results of Boas suggest a uniformity of type with a majority of the Plains tribes (Blackfoot, Cheyenne, Arapaho, Crow, Pawnee) and less certainly with the Micmac, Abenaki, Delaware, Iroquois, Ottawa, and Menomini.

The half-bloods in the series are for the most part the results of the intermarriage of French, Scotch, Irish, and English men with Indian women. In the study of skin, hair, and eye color the tests were not sufficiently sensitive to bring out any clear cut and certain differences between the full-bloods and half-bloods, if such differences exist. In all of these characters the half-bloods seem to stand very close to the full-bloods. In regard to the amount of hair on the face (beard and moustache) the half-bloods stand intermediate between the Indians and whites.

The anthropometric characters bring out two points of interest: First, that in general body form and proportions the Sioux Indians are not very different from the whites with whom they have mixed. There are practically no differences between the full-bloods and half-bloods in absolute or relative shoulder height, shoulder width, sitting height, arm length, arm reach, and very small differences in the cephalic, facial, and nasal indices. Second, by far the most noticeable and consistent differences are differences in absolute size. The half-bloods are taller than the full-bloods. On the other hand, the full-bloods have the more massive heads, faces, and noses. While the relation of these diameters, as expressed by indices, are very much alike, the absolute diameters are different. The most marked difference is in the width of face. The full-bloods have a much wider face than the half-bloods or whites. The height of the face of full-bloods is also greater and the area of the face much larger than in the half-bloods. It seems that in all those characters in which the Indian differs most markedly from the whites, the half-bloods stand nearer to the Indians than to the whites.

In our present investigation we found the half-bloods more variable than the full-bloods in stature, shoulder height, sitting height, head length, face width, cephalo-facial index, facial index and nasal index. In the remaining twelve of the twenty-one observations the full-bloods were slightly more variable. But in nearly every seriation the distribution among the half-bloods was more irregular.

In our correlations we found the closest relationships to exist between diameters in the same axis such as stature and arm reach, stature and arm length, stature and sitting height, and width of head and width of face. A fair degree of correlation exists between gross diameters in opposite

axes such as stature and width of shoulder. Other diameters in different axes, such as length and width of head, height and width of face, and height and width of nose show a lower degree of correlation. The differences in degree of correlation of two diameters between full-bloods and half-bloods are not very great. On the whole, it does not seem as if intermixture had seriously affected the degree of correlation. In the instance of face width and head width the correlation seems to be increased in the half-bloods.

The distribution of face width among the one-fourth Indians, two-fourths Indians, one-half Indians, and three-fourths Indians indicates that the inheritance is alternating. The mixed individual tends to inherit either the Indian or European type of face, intermediate types being of relatively rare occurrence. A high degree of correlation between width of face and width of head was found in the mixed group. Individuals with a narrow head had a tendency to inherit also the narrow face and vice versa. A summary of the anthropometric results will be found in table I and of the correlations in table II.

A detailed report will be published in the *Anthropological Papers of the American Museum of Natural History*, vol. 23, part 3.

TABLE I

SUMMARY OF THE ANTHROPOMETRIC RESULTS

MALE ADULT FULL-BLOOD AND HALF-BLOOD SIOUX INDIANS

CHARACTER	AVERAGE		STANDARD DEVIATION (σ)		COEFFICIENT OF VARIABILITY (V)	
	Full-bloods	Half-bloods	Full-bloods	Half-bloods	Full-bloods	Half-bloods
Number of Cases	540	77	540	77	540	77
Stature	172.4	173.5	5.64	6.81	3.27	3.92
Shoulder Height	142.7	142.3	5.03	6.07	3.52	4.26
Shoulder Width	38.8	38.9	1.92	1.89	4.94	4.83
Index Shoulder Width	22.5	22.4	1.10	1.01	4.88	4.51
Sitting Height	88.5	89.6	3.50	4.39	3.95	4.89
Index Sitting Height	51.4	51.6	1.68	1.94	3.26	3.76
Arm Reach	181.4	182.2	7.03	6.99	3.87	3.83
Index Arm Reach	105.2	105.0	2.41	2.19	2.29	2.09
Arm Length	77.0	77.3	3.57	3.28	4.64	4.24
Index Arm Length	44.6	44.6	1.47	1.26	3.29	2.82
Head Length	194.9	194.4	6.16	7.12	3.16	3.66
Head Width	155.1	154.3	5.39	5.04	3.47	3.26
Cephalic Index	79.6	79.4	3.20	2.64	4.03	3.33
Face Width	149.1	143.4	5.45	5.49	3.65	3.83
Cephalo-Facial Index	96.1	92.9	3.22	3.23	3.35	3.48
Anatomical Face Height	124.6	121.5	6.39	6.36	5.12	5.23
Anatomical Face Index	83.6	84.8	4.84	5.28	5.78	6.22
Nasal Height	58.3	54.9	3.94	3.55	6.75	6.48
Nasal Width	39.9	37.6	3.22	3.04	8.07	8.08
Nasal Index	68.8	69.2	7.05	7.08	7.05	7.08

TABLE 2

CORRELATIONS IN THE MALE ADULT SERIES

CORRELATED MEASUREMENTS	COEFFICIENTS OF CORRELATION	
	540 Full-bloods	77 Half-bloods
Stature and Sitting Height	0.61	0.65
Stature and Width of Shoulder	0.35	0.48
Stature and Arm Reach	0.81	0.85
Stature and Length of Arm	0.70	0.76
Length and Width of Head	0.27	0.54
Width of Head and Width of Face	0.55	0.51
Anatomical Height and Width of Face	0.16	0.08
Height and Width of Nose	0.05	0.02

TRANSMISSION OF EYE-DEFECTS INDUCED IN RABBITS BY MEANS OF LENS-SENSITIZED FOWL-SERUM

BY M. F. GUYER AND E. A. SMITH

ZOÖLOGICAL LABORATORY, UNIVERSITY OF WISCONSIN

Communicated by L. Hektoen, January 19, 1920

As the work progresses it is becoming increasingly evident that the discoveries made in the field of serology all have their broader biological aspects, and that they afford new methods of attacking certain fundamental biological problems. Not the least of these is a possible method of breaching the wall which has gradually come to surround the long-standing problem of provoking specific modifications in the germ-cell through the direct action of external agencies, or indirectly, through changes produced in the parental body. For if external influences *can* be transmitted to the germ-cell, the one obvious means of conveyance in higher animals is the blood, and when one considers the protean possibilities which modern work has revealed in the blood, it is certainly a rational quest to seek in this medium a possible means of altering the germ. To set the problem more specifically, if a serum of one species of animal can be so sensitized to a given tissue or tissues of another species that it will become toxic or lytic for the tissue in question, may it not be that there is sufficient constitutional identity between the mature substance of the tissue and at least some of its material antecedents in the germ, that the latter may also be influenced specifically by the sensitized serum? Or may not changes in its own tissues originate antibodies in the blood serum of a given individual which will not only react with the tissue elements themselves but with their correlatives in the germ cells?

In an attempt to find answers to these and kindred questions we have been engaged in series of experiments which have extended over a period

of four years and which are still in progress. To state briefly some of our results, we have been able occasionally to produce what appear to be specific antenatal lens-defects in rabbits and mice by means of fowl-serum sensitized to rabbit-lens and mouse-lens respectively. And, what is probably of more interest, breeding experiments with the affected rabbits show that the defect, once established, may be transmitted to subsequent generations.

Since most of the work was done with rabbits the following account is concerned with them alone. Chickens were employed as the source of the antibodies. To produce these, the fresh lenses of rabbits were pulped thoroughly in a mortar and diluted with normal salt solution. By means of a hypodermic syringe about four cubic centimeters of the emulsion was injected into the peritoneal cavity of the fowl. In later experiments some of the fowls were injected intravenously as well as intraperitoneally. Ordinarily the fowls were treated with such lens-emulsions at intervals of a week for four or five weeks and were then allowed to rest a week of ten days before bleeding.

In the mean time rabbits had been bred so as to have the young advanced to about the tenth day of pregnancy. From four to seven cubic centimeters of the sensitized fowl-serum (sometimes diluted with normal saline) were injected into the pregnant rabbits at intervals of two or three days from about the tenth to the twentieth days of pregnancy. Some rabbits died from the treatment, many young were killed in utero, but a goodly number also survived. Among the latter were occasional individuals with opaque (sometimes liquid) lenses and eyes otherwise defective (reduced size, cleft iris, persistent hyaloid artery, bluish or silvery color instead of the characteristic red of the albino eye, and occasionally almost complete disappearance of the eyeball). Taking into account the method of embryological development of the eye, however, these defects are probably all attributable to the early injury of the lens.

Out of sixty-one surviving young from mothers treated during pregnancy with lens-sensitized serum, four had one or both eyes conspicuously defective and five others had eyes which, though less noticeably modified, were nevertheless abnormal. Others also may have had their eyes affected, but if so, the condition was not observable from the exterior. As a matter of fact, liquid lenses were found in a few young ones which were dissected in the earliest experiments. This condition was not detectable in the living animal. Of the controls, from mothers treated with pure (i. e., unsensitized) fowl-serum twelve living young were obtained, and from others treated with serum sensitized to rabbit-tissue other than lens, thirty-six young were secured, or a total of forty-eight which survived long enough to show the condition of their eyes. In not one of these forty-eight controls was there evidence of eye-defect. The experiments indicate, therefore, that the effect of the lens-sensitized serum is specific.

As already mentioned, once the defect is secured, it may be transmitted to subsequent generations through breeding. So far we have succeeded in passing it to the sixth generation. There seems to be no reason why it will not go on indefinitely since the imperfection tends to become worse in succeeding generations and also to occur in a proportionally greater number of young. The transmission was not infrequently of an irregular unilateral type, sometimes only the right, at others only the left eye showing the defect. In later generations there was an increasing number of young which had both eyes affected.

To meet the objection that we were not getting instances of true inheritance in each generation but merely placental transmissions of antibodies or kindred substances from the blood stream of the mother, it was obviously necessary to establish the descent through the male line alone. To do this females from strains of rabbits unrelated to our defective-eyed stock were mated to defective-eyed males. The first generation produced in this way were invariably normal-eyed, but when females of this generation in turn were mated to defective-eyed males the defect reappeared in some of the progeny somewhat after the manner of an extracted Mendelian recessive. Inasmuch as the defect can thus be made to reappear in the descendants of a male with abnormal eyes when he is mated to a female from unrelated and untreated stock, it is obvious that it could only have been conveyed through the germ-cells of the male, and that it may, therefore, be pronounced an example of true inheritance.

The detailed study, with charts, pedigrees, drawings and photographs will appear shortly in the *Journal of Experimental Zoölogy*.

ON THE MECHANISM OF FEVER REDUCTION BY DRUGS

By Henry G. Barbour and Julian B. Herrmann

Department of Pharmacology, Yale University, School of Medicine

Communicated by L. B. Mendel, Read before the Academy, November 10, 1919

Correlation of antipyretic drug action with the carbohydrate metabolism has been suggested by one of us[1] as a result of two findings in fever patients. In the first place dextrose by mouth has been found frequently to exert a mild antipyretic action; on the other hand, acetyl-salicylic acid or antipyrine under similar conditions increase the respiratory quotient, thus indicating a relatively augmented carbohydrate combustion. These facts point strongly to a mobilization of sugar by such drugs.

We have, therefore, investigated the effects of antipyretics upon normal and fevered dogs with particular attention to the concentration of dextrose in the blood. Mild fever was produced by the subcutaneous injection upon the preceding day of a suspension of killed colon bacilli. Doses of from 0.2 to 0.5 gram. per kilo of the following drugs were given: sodium salicylate, quinine hydrochloride, antipyrine (all subcutaneously)

and acetyl-salicylic acid (per os). In fevered dogs a fall in body temperature of about 1° C. resulted; on the other hand a slight increase was the only observed change in the temperature of normal animals.

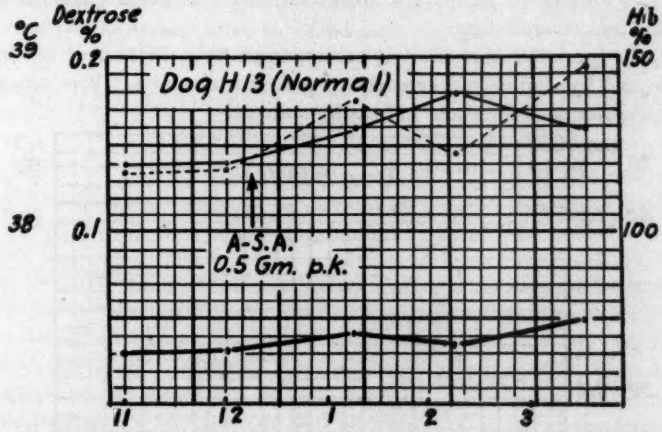


FIG. 1. Effects of acetyl-salicylic acid (0.5 gm. per kilo) on normal dog. Rectal temperature: light line; blood dextrose per cent: broken line; hemoglobin per cent: heavy line.

Blood Dextrose.—Both series of dogs were subjected to investigation of the blood sugar concentration by the Benedict-Lewis method. In all animals irrespective of the fevered condition there resulted a moderate hyperglycaemia, the extent of which may be judged from the following summary:

DEXTROSE CONCENTRATION IN THE BLOOD

	BEFORE ANTIPYRETIC %	MAXIMUM AFTER ANTIPYRETIC %
Average of 13 normal dogs	0.137	0.186
Average of 10 fevered dogs	0.139	0.218

Blood Volume.—The hemoglobin concentration was followed in most of these experiments simultaneously with the blood sugar, employing the colorimetric method of Cohen and Smith.[2] In the normal dogs was found a rise in the hemoglobin percentage accompanying the slight temperature increase. The thickening of the blood thus indicated was insignificant in comparison with the observed dextrose increase, the latter being thus proven absolute rather than merely relative.

Fevered dogs, on the other hand, showed a distinctly increased blood volume as indicated by the diminished hemoglobin percentage accompanying the antipyretic effect. Dilution of the blood of fever patients by acetyl-salicylic acid and by antipyrine has also been observed by one of us.

The above described effects as exemplified in normal and fevered dogs are illustrated in figures I and II, respectively. The following conclusions have been drawn from this work:

(1) A variety of antipyretic drugs increase the blood sugar in both normal and fevered dogs.

(2) In the latter this effect is accompanied by a dilution of the blood (indicated by diminished hemoglobin percentage) and a fall in temperature, neither of which occur in healthy animals.

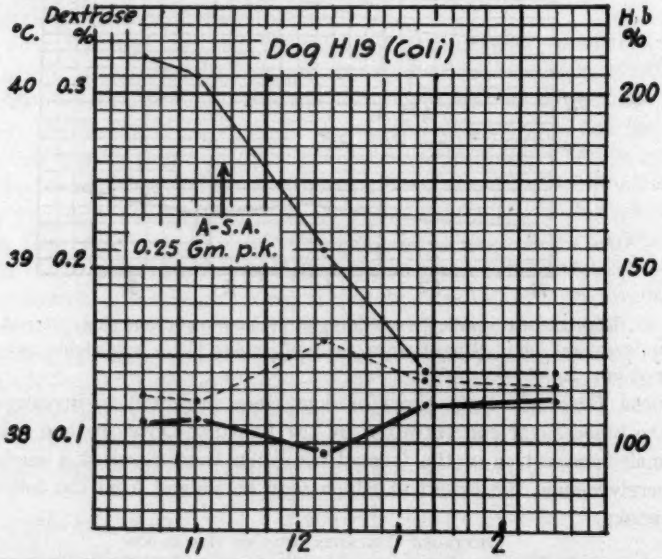


FIG. 2. Effects of acetyl-salicylic acid (0.25 gm. per kilo) on dog treated on preceding day with suspension of killed colon bacilli. (See Fig. 1.)

(3) *Theory of the mechanism of fever reduction by drugs.*—Antipyretic drugs increase the blood content of dextrose, a substance itself often exhibiting temperature-reducing properties when introduced from without. Moderate hyperglycaemia favors dilution of the blood.[3] In fevered animals antipyretics actually produce such a plethora, the hyperglycaemia probably contributing largely to this effect. Plethora promotes dissipation of heat both by radiation (peripheral vaso-dilation) and water evaporation from the surface of the body. The occurrence of plethora with its resulting anti-pyretic effect is apparently limited to fevered animals. This fact should probably be attributed not so much to a greater degree of hyperglycaemia as to the relative water retention by the tissues which is said to accompany febrile conditions.[4]

[1] Barbour, H. G., "Antipyretics" (Papers I and III), *Arch. Int. Med.*, Dec., 1919; and *Proc. Soc. Exp. Biol. Med.*, **16**, 1919 (136).

[2] Cohen, B. and Smith, A. H., *J. Biol. Chem.*, **39**, 1919 (489).

[3] Cf. Fisher, G. and Wishart, M. B., *J. Biol. Chem.*, **13**, 1912 (49).

[4] A part of the expenses of the work herein reported has been defrayed from the Francis E. Loomis Research Fund of the Yale University School of Medicine. More extensive reports will be sent to the *Journal of Pharmacology & Experimental Therapeutics* and to the *Archives of Internal Medicine*.

INFERENCES FROM THE HYPOTHESIS OF DUAL ELECTRIC CONDUCTION; THE THOMSON EFFECT

BY EDWIN H. HALL

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Communicated January 29, 1920

At the Washington meeting of the National Academy of Sciences in April, 1919, I presented two papers that have not yet been published. One was on the *Effect of Pressure on Electric Resistance and on Peltier Heat in Metals*, the other on *Thermal Conduction in Metals*, both being written from the standpoint of Dual Electric Conduction.

The first named of these two papers contained implicitly the following propositions:

1*a*. Increase of pressure should, by bringing the atoms and the metal ions closer together, increase k_a, the associated-electron conductivity, and decrease k_f, the free-electron conductivity. We might, then, expect the total conductivity, k, to increase under pressure in metals having a relatively small value of $(k_f \div k_a)$ and to decrease in metals having a relatively large value of this ratio.

1*b*. As antimony and bismuth have exceptionally small values of k, they probably have exceptionally large values of $(k_f \div k_a)$, and this may account for the fact that, among twenty metals examined by Bridgman, these two were the only ones to show a decrease of conductivity under an increase of pressure.

2*a*. If the ratio $(k_f \div k_a)$ is greater in metal B than in metal A, ionization must occur at the junction of the two metals when a current flows from A to B, and re-association must occur there when the current flows from B to A. As ionization is doubtless accompanied by absorption of heat and re-association by evolution of heat, we have here an action which may play a very important part, if not the chief part, in the Peltier effect.

2*b*. The exceptionally large value of $(k_f \div k_a)$ that probably exists in bismuth may account for the fact that heat is absorbed when a negative current goes into this metal from any other.

2*c*. As increase of pressure probably decreases the ratio $(k_f \div k_a)$, we should expect an absorption of heat where a negative current flows from a metal under high pressure to the same metal uncompressed. If we call this effect of compression *plus* and the opposite effect *minus*, we

find that, among eighteen metals examined by Bridgman under high pressures, thirteen showed *plus* effects only, none showed *minus* effects only, but five showed mixed effects, *minus* at 0° C. and *plus* at 100° C.

3*a*. According to the theory under discussion the Thomson effect should disappear if $(k_f \div k_a)$ became zero. Accordingly increase of pressure, causing a decrease of this ratio, should diminish the Thomson effect. Among eighteen metals examined by Bridgman under pressure, nine showed a decrease throughout the whole range of observation, one showed an increase through the whole range of observation, one showed zero change everywhere, the other seven showed mixed effects.

The following may be taken as a summary of the second paper:

1. Thermal conduction in a metal may be due to the convective action of a circulating electric current, free electrons moving down the temperature gradient and associated electrons moving up, with ionization at the hot end of the metal, involving absorption of heat, and re-association at the cold end, involving the emission of heat.

2. A quantitative test of this thesis, made necessarily with various assumptions, indicated ionizing potentials of the same order of magnitude as those observed in the ionization of metal vapors; but it seemed doubtful whether values so large as those here indicated are consistent with the magnitudes of the Thomson effect. It is to be noted that ionization within the solid body of a metal may well require less energy than ionization of the vapor.

It seemed desirable to study the Thomson effect more fully from the standpoint of dual electric conduction before proceeding farther with the theory of thermal conduction. The results of an examination of the Thomson effect are now ready for publication.

THE THOMSON EFFECT

In what follows:

n = the no. of free electrons per cu. cm. of a metal.

v = the no. of cu. cm. of metal containing 1 gm. of free electrons.

m = mass of electron, and G = no. of electrons in 1 gm. of electrons.

Then $nv = G = 1 \div m$. (1)

p = press. of free electrons, in dynes per sq. cm. of cross-section of the metal.

R = the gas constant for one molecule, or for one electron, $= 1.37 \times 10^{-16}$.

Then $p = nRT$, (2)

and, for 1 gm. of free electrons, $pv = nvRT = RT \div m$. (3)

P = electro-mag. pot. due to purely electric forces within the metal.

P_a = electro-mag. pot. due to differential attraction of the unequally heated metal for the associated electrons.

P_f = corres. pot. due to differential attraction of the metal for the free electrons.

k_a = part of specific electric conductivity due to associated electrons.
k_f = part of specific electric conductivity due to free electrons.
k = $k_a + k_f$ = total electric conductivity.
e = the electron charge in electro-mag. measure = 1.6×10^{-20}.
λ = the latent heat of ionization per $(1 \div e)$ electrons, in ergs.
λ' = the latent heat of ionization per electron, in ergs.

Hypothesis (A) is, that the mere mechanical tendency of the free electrons is toward uniformity of pressure throughout the unequally heated metal.

Hypothesis (B), alternative with (A), is that the mechanical tendency of the free electrons is toward the state of equilibrium produced by thermal effusion; that is,

$$p \div T^{1/2} = \text{constant.} \qquad (4)$$

Let C of figure (1) be the cold end and H the hot end of a metal bar forming part of a circuit in which an electric current is maintained by thermo-electric action, the resistance of some part of the circuit being so great that the conditions existing in CH are very little different from those of equilibrium. In this case the Joule heat generated in CH can be neglected in comparison with the Thomson heat there generated or absorbed.

Contrary to custom, the direction of the stream of electrons through the metal will, in this paper, be taken as the direction of the current, and accordingly σ, the Thomson heat at any temperature T, will be defined as the heat absorbed by the electromagnetic unit quantity of electricity, $(1 \div e)$ electrons, in going through the metal from a place of temperature $(T - 0.5)$ degree to a place of $(T + 0.5)$ degree. This definition will make σ negative for copper and positive for iron. The value of σ will be expressed in ergs.

FIG. 1

When the unit quantity of electricity, $(1 \div e)$ electrons, $(m \div e)$ gm., goes through the slice dl of the bar CH, from the isothermal surface T to the isothermal surface $T + dT$, the fraction $(k_f \div k)$ of it consists of free electrons and the part $(k_a \div k)$ of associated electrons. We have now to take note of the changes of energy, of various kinds, undergone by these two parts of the current. We shall list the various forms of energy here considered under five general heads: (1) bulk potential energy, or pv potential energy, to which the free electrons only are subject; (2) kinetic energy of the electrons, which we shall regard as negligible in the associated electrons and equal to that of monatomic gas molecules in the free electrons; (3) electric-charge potential energy, the P energy, to which both the free and associated electrons are alike subject; (4) the P_f potential

energy, which affects free electrons only; (5) the P_a potential energy, which affects the associated electrons only. We must take account also of the gain of energy involved in the ionization which may occur in the current from T to $T + dT$, due to the increase in the ratio $(k_f \div k)$ with rise of temperature.

Under hypothesis (A): All changes due to change of $(k_f \div k)$ being considered last, we have as the change of pv energy, (see equation (3))

$$(k_f \div k)(m \div e)d(pv) = (k_f \div k)(R \div e)dT. \qquad (a)$$

The gain of thermal kinetic energy by the free electrons is

$$(k_f \div k)(1 \div e) \cdot \frac{3}{2} RdT. \qquad (b)$$

The gain of P potential energy is dP. (c)

The gain of P_f potential energy is $(k_f \div k)dP_f$. (d)

The gain of P_a potential energy is $(k_a \div k)dP_a$ (e)

The gain of energy through ionization is

$$\lambda d(k_f \div k) = (1 \div e)\lambda' d(k_f \div k). \qquad (f)$$

The sum of all these quantities is the Thomson-effect heat absorbed between T and $T + dT$; that is

$$\sigma dT = (a) + (b) + (c) + (d) + (e) + (f) \qquad (5)$$

From this we get

$$\sigma = \frac{k_f}{k} \cdot \frac{5}{2}\frac{R}{e} + \left[\frac{k_f}{k} \cdot \frac{dP_f}{dT} + \frac{k_a}{k} \cdot \frac{dP_a}{dT} + \frac{dP}{dT}\right] + \frac{\lambda'}{e} \cdot \frac{d(k_f \div k)}{dT}. \qquad (6)$$

From the conditions of equilibrium, under hypothesis (A), in a detached bar like CH we have[1]

$$\frac{k_f}{k} \cdot \frac{1}{e}\left(R + \frac{RT}{n} \cdot \frac{dn}{dT}\right) + \left[\frac{k_f}{k} \cdot \frac{dP_f}{dT} + \frac{k_a}{k} \cdot \frac{dP_a}{dT} + \frac{dP}{dT}\right] = 0 \qquad (7)$$

Subtracting (7) from (6) we get

$$\sigma = \frac{k_f}{k} \cdot \frac{3}{2}\frac{R}{e} - \frac{k_f}{k} \cdot \frac{RT}{ne} \cdot \frac{dn}{dT} + \frac{\lambda'}{e} \cdot \frac{d(k_f \div k)}{dT}. \qquad (8)$$

Under hypothesis (B) we get $0.5R$ instead of the first R in (7), and this gives $2R$ instead of $1.5R$ in the first term of (8). Hypothesis (B) makes no other change in (7) or (8).

I shall now try to put equation (8) into a form suitable for dealing with the values of σ found by Bridgman[2] in his experiments on a large number of metals. In this undertaking I assume that for present purposes the following equations hold above 0° centigrade:

$$n = zT^q, \qquad (9)$$

and

$$(k_f \div k) = C + C_1 t + C_2 t^2, \qquad (10)$$

where z, q, C, C_1 and C_2 are constants, and t is temperature on the ordinary centigrade scale.

I shall, moreover, assuming that the total heat of ionization per electron is made up of a part λ'_0; due to the overcoming of atomic attraction, a part $1.5RT$ for the kinetic energy gained, and a part RT for the pv potential energy gained, write

$$\lambda' = \lambda'_0 + 2.5RT = \lambda'_0 + 2.5R(273 + t). \quad (11)$$

Keeping to hypothesis (A), and so using equation (8) for σ, I get by substitution according to eqs. (9), (10) and (11),

$$\sigma = K + (K_1 + K_2 t)T, \quad (12)$$

where K, K_1 and K_2 are constants,[3] defined by the equations

$$K = \frac{R}{e}\left[C(1.5-q) + C_1\left(\frac{\lambda'_0}{R} - 273(1.5-q)\right) - 273C_2\left(\frac{2\lambda'_0}{R} - 273(1.5-q)\right)\right] \quad (13)$$

$$K_1 = \frac{R}{e}\left[C_1(4-q) + .C_2\left(\frac{2\lambda'_0}{R} - 273(1.5-q)\right)\right] \quad (14)$$

$$K_2 = \frac{R}{e}.C_2(6.5-q). \quad (15)$$

I have put σ into the form shown by equation (12) in order to make my expression for it correspond as nearly as may be to that used by Bridgman to set forth the results of his experiments. He writes, in substance,

$$\sigma = (A + Bt)T, \quad (16)$$

where A and B are constants, the latter being zero in many metals.

Bridgman finds nothing corresponding to my constant K, and I have spent much labor in attempting to get rid of this constant; but no reasonable assumption that I can make eliminates it from my general expression for σ. On the other hand, equation (13) shows that K is the sum of many terms, some positive, some negative, and there is nothing to show that it may not be very small, too small to appear in such experiments as those of Bridgman. Accordingly, in dealing with his observations I put K, and so the second member of (13), equal to zero. This gives me an equation of which I make frequent use in the form

$$C = \left[-C_1\left(\frac{\lambda'_0}{R} - 273(1.5-q)\right) + 273C_2\left(\frac{2\lambda'_0}{R} - 273(1.5-q)\right)\right] \div (1.5-q). \quad (17)$$

As to the A and B of equation (16), I take these to be, respectively, the K_1 and the K_2 of my equations, and, as Bridgman gives the value of A and B for every case dealt with, I have the K, K_1 and K_2, of equations (13), (14) and (15), replaced by definite numerical terms.

These three equations now contain the five unknowns, C, C_1, C_2, λ'_0, and q. Accordingly, I must assume values for two of these quantities in order to evaluate the other three. As a rule, I have assumed values of

q and λ'_0, different ones in succession for each metal, and have then worked out the corresponding values of the other quantities, following the order C_2, C_1, C.

Values of q and λ'_0 that would lead to values for C greater than 1 or less than 0 are of course rejected, as such values of C would be meaningless, but negative values of C_1 and C_2 are not to be regarded as impossible. I was at first inclined to the opinion that the ratio $(k_f \div k)$ would always increase with rise of temperature, but this is not a logical necessity, in the present state of our knowledge, and it appears from what follows that the ratio in question is quite as likely to decrease as to increase in the temperature ascent from 0° to 100°. An interesting relation between this conclusion and the observations of Bridgman on change of resistance under pressure, at various temperatures, will be shown in this paper.

Cases in which K_2, or B, is 0: In twelve of the seventeen metals for which Bridgman gives the value of σ the B of equation (16) is zero. Such cases are very easy to deal with. We have $K_2 = 0$, and so, from equation (15), $C_2 = 0$, unless q has the improbably large value 6.5. If C_2 is 0, we have, from equation (14),

$$C_1 = K_1 \text{ (or } A) \frac{e}{R} \div (4 - q). \tag{18}$$

Substituting for C_1 in equation (17), we get

$$C = \left[-K_1 \cdot \frac{e}{R} \left(\frac{\lambda'_0}{R} - 273(1.5 - q) \right) \div (4 - q) \right] \div (1.5 - q). \tag{19}$$

For any given value of q this becomes

$$C = K' \lambda'_0 - K'' \tag{20}$$

where K' and K'' are new constants, the values of which depend on q.

This equation shows that, for a fixed value of q, we can represent the

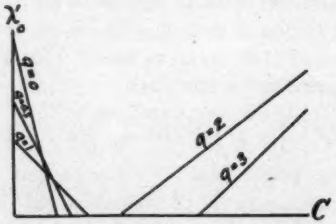

FIG. 2

relation of C to λ'_0 by means of a straight line drawn on the $(C\text{-}\lambda'_0)$ plane. Such a line is useful for purposes of interpolation and extrapolation.

It is to be noted that the metals for which K_2 is 0 fall into two groups, for one of which K_1 is positive, while for the other it is negative.

Figure 2 shows the general character of the set of q-constant lines for the first group, and figure 3 does the same for the second group.

For both of these groups 1.5 is a critical value for q. Examination of equation (19) shows that, when $q = 1.5$, C becomes infinite unless λ'_o at the same time becomes 0; and if λ'_o becomes 0 while $q = 1.5$, C becomes

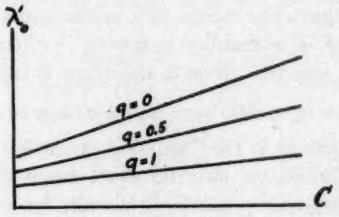

FIG. 3

indeterminate. In both figure 2 and figure 3, therefore, $q = 1.5$ would imply a line coincident with the C axis.

If q in equation (19) has a value between 1.5 and 4, C will have the same sign as K_1, which is positive for the first group and negative for the second group. Accordingly, since negative values of C are meaningless, $1.5 < q < 4$ is possible for the first group but not for the second group.

Cases in which K_2 is not o. There are in Bridgman's list five metals for which K_2, or B, is not zero. These are aluminium, gold, iron, molybdenum, and thallium. Finding the value of C, for a given combination of q and λ'_o, in these metals, is a somewhat roundabout, though not difficult, process. The value of C_2 is found by use of equation (15), then the value of C_1 by use of (14), then the value of C by use of (17). The q-constant lines on the $(C\text{-}\lambda'_o)$ plane are no longer straight, as they are in figures 2 and 3. Figure 4 shows their general shape for $q = 0$, $q = 0.5$ and $q = 1$, in the range from $C = 0$ to $C = 0.20$. It is to be noted that, though K_1 is positive in iron and thallium while negative in aluminium, gold,

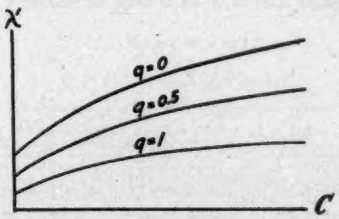

FIG. 4

and molybdenum, the lines in question are of the same general shape and arrangement for all five metals. The value of K_2 is positive for all.

Examination of equation (17) shows that, when $q = 1.5$, C is ∞ or is indeterminate. In the latter case, we may have either $\lambda'_o = 0$ or $C_1 = 546C_2$.

When q is made larger than 1.5 but smaller than 6.5, negative values of C result, in all five of the metals, so far as I have examined the matter.

The following tables represent my results for all of the elementary metals for which Bridgman gave the values of σ in the paper already referred to. A dash (—) in place of a number indicates that the number would be negative, and that a negative value in this place is regarded as impossible.

The values of $(k_f \div k)$ at 100° are found by use of equation (10).

The δ_0 of these tables is the "ionizing potential" needed for dealing with the attraction which an electron must overcome in the process of ionization. If this ionizing potential is 1 volt, for example, the *internal* work of ionization is about 11700 R ergs per electron. The *external* work of ionization, to provide the kinetic energy and the pv energy of the gaseous state, is 2.5 RT ergs per electron. This at 0° would require an ionizing potential of about 0.058 volt, which must be added to the δ_0 of the tables in order to get the total ionizing potential in volts.

The general significance of these tables can be illustrated as follows, with numbers taken from table 1: If in cobalt the value of C is 30%, the Thomson effect heat, as found by Bridgman for this metal, can be accounted for either by

taking $q = 2$, $\delta_0 = 0.02$, $C_1 = 457 \times 10^{-6}$, $C_2 = 0$,

or by

taking $q = 3$, $\delta_0 = 0.01$, $C_1 = 915 \times 10^{-6}$, $C_2 = 0$.

In the first case $(k_f \div k)$ will be 0.346 at 100°, while in the second case it will be 0.392. If the value of C is 40%, there are corresponding values of q, δ_0, etc., that will account for σ. A like statement would hold for any value of C between 30% and 40%, and for numberless other values of this ratio.

First Group: Metals for which $K_1 > 0$ and $K_2 = 0$

TABLE 1: COBALT.

$K_1 = 7.8$, $K_2 = 0$, $C_2 = 0$

C, or $(k_f \div k)$, at 0°	$C_1 = 228 \times 10^{-6}$			$C_1 = 261 \times 10^{-6}$			$C_1 = 305 \times 10^{-6}$			$C_1 = 457 \times 10^{-6}$			$C_1 = 915 \times 10^{-6}$		
	q	δ_0	$(k_f \div k)$ at 100°	q	δ_0	$(k_f \div k)$ at 100°	q	δ_0	$(k_f \div k)$ at 100°	q	δ_0	$(k_f \div k)$ at 100°	q	δ_0	$(k_f \div k)$ at 100°
0.01	0	.03	.033	.5	.02	.036	1	.01	.041	2	—		3	—	
0.08[1]	0	—		.5	—		1	.00	.111	2	—		3	—	
0.20	0	—		.5	—		1	—		2	.01	.246	3	—	
0.30	0	—		.5	—		1	—		2	.02	.346	3	.01	.392
0.40	0	—		.5	—		1	—		2	.03	.446	3	.02	.492

[1] If there is any combination of q and δ_0 that will make $C = 0.10$ for cobalt, the q must be very near 1.5.

TABLE 2: NICKEL.

$K_1 = 3.56$, $K_2 = 0$, $C_2 = 0$

C, or $(kf+k)$, at 0°	$C_1 = 104 \times 10^{-6}$			$C_1 = 119 \times 10^{-6}$			$C_1 = 139 \times 10^{-6}$			$C_1 = 208 \times 10^{-6}$			$C_1 = 416 \times 10^{-6}$		
	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°
0.01	0	.02	.020	.5	.016	.022	1	.01	.024	2	—		3	—	
0.10	0	—		.5		—	1	—		2	.010	.121	3	—	
0.20	0	—		.5		—	1	—		2	.033	.221	3	.03	.242
0.30	0	—		.5		—	1	—		2	.056	.321	3	.06	.342
0.40	0	—		.5		—	1	—		2	.079	.421	3	.09	.442

PALLADIUM

$K_1 = 3.52$, $K_2 = 0$, $C_2 = 0$

Palladium is so like nickel in its Thomson effect that table 2 will serve for it.

TABLE 3: PLATINUM.

$K_1 = 2.67$, $K_2 = 0$, $C_2 = 0$

C, or $(kf+k)$, at 0°	$C_1 = 78 \times 10^{-6}$			$C_1 = 89 \times 10^{-6}$			$C_1 = 104 \times 10^{-6}$			$C_1 = 156 \times 10^{-6}$			$C_1 = 312 \times 10^{-6}$		
	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°
0.01	0	.02	.018	.5	.01	.019	1	.01	0.20	2	—		3	—	
0.10	0	—		.5	—		1	—		2	.02	.116	3	.01	.131
0.20	0	—		.5	—		1	—		2	.05	.216	3	.05	.231
0.30	0	—		.5	—		1	—		2	.08	.316	3	.09	.331
0.40	0	—		.5	—		1	—		2	.11	.416	3	.13	.431

TABLE 4: TIN.

$K_1 = 0.134$, $K_2 = 0$, $C_2 = 0$

C, or $(kf+k)$, at 0°	$C_1 = 392 \times 10^{-6}$			$C_1 = 448 \times 10^{-6}$			$C_1 = 520 \times 10^{-6}$			$C_1 = 780 \times 10^{-6}$			$C_1 = 1560 \times 10^{-6}$		
	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°
0.01	0	—		.5	—		1	—		2	0.05	.011	3	0.05	.012
0.10	0	—		.5	—		1	—		2	0.52	.101	3	0.79	.102
0.20	0	—		.5	—		1	—		2	1.04	.201	3	1.62	.202
0.30	0	—		.5	—		1	—		2	1.56	.301	3	2.45	.302
0.40	0	—		.5	—		1	—		2	2.08	.401	3	3.28	.402

TABLE 5: ZINC.

$K_1 = 0.99, K_2 = 0, C_2 = 0$

C, or $(kf+k)$, at 0°	$C_1 = 29 \times 10^{-6}$			$C_1 = 33 \times 10^{-6}$			$C_1 = 38 \times 10^{-6}$			$C_1 = 58 \times 10^{-6}$			$C_1 = 116 \times 10^{-6}$		
	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°
0.01	0	—		.5	—		1	.0	.014	2	—		3	—	
0.10	0	—		.5	—		1	—		2	.06	.106	3	.07	.112
0.20	0	—		.5	—		1	—		2	.13	.206	3	.18	.212
0.30	0	—		.5	—		1	—		2	.20	.306	3	.30	.312
0.40	0	—		.5	—		1	—		2	.28	.406	3	.41	.412

Second Group: Metals for which $K_1 < 0$ and $K_2 = 0$

TABLE 6: BISMUTH.

$K_1 = -3.2, K_2 = 0, C_2 = 0$

C, or $(kf+k)$, at 0°	$C_1 = -94 \times 10^{-6}$			$C_1 = -107 \times 10^{-6}$			$C_1 = -125 \times 10^{-6}$		
	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°
0.01	0	0.05	0.001	0.5	0.03	0.001	1	0.01	0.000
0.10	0	0.17	0.091	0.5	0.10	0.089	1	0.04	0.087
0.20	0	0.31	0.191	0.5	0.18	0.189	1	0.08	0.187
0.40	0	0.58	0.391	0.5	0.34	0.389	1	0.14	0.387
0.60	0	0.85	0.591	0.5	0.50	0.589	1	0.21	0.587

TABLE 7: CADMIUM.

$K_1 = -32.4, K_2 = 0, C_2 = 0$

C, or $(kf+k)$, at 0°	$C_1 = -94 \times 10^{-6}$			$C_1 = -107 \times 10^{-6}$			$C_1 = -125 \times 10^{-6}$		
	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°
0.01	0	0.036		0.5	0.024	—	1	0.012	—
0.10	0	0.049	0.006	0.5	0.032	—	1	0.015	—
0.20	0	0.062	0.106	0.5	0.040	0.093	1	0.019	0.075
0.40	0	0.090	0.306	0.5	0.055	0.293	1	0.026	0.275

TABLE 8: COPPER.

$K_1 = -0.966, K_2 = 0, C_2 = 0$

C, or $(kf+k)$, at 0°	$C_1 = -28 \times 10^{-6}$			$C_1 = -35 \times 10^{-6}$			$C_1 = -38 \times 10^{-6}$		
	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100
0.01	0	0.08	0.007	0.5	0.05	0.006	1	0.02	0.006
0.10	0	0.49	0.097	0.5	0.26	0.096	1	0.12	0.096
0.20	0	0.95	0.197	0.5	0.48	0.196	1	0.24	0.196
0.40	0	1.87	0.397	0.5	0.92	0.396	1	0.48	0.396

TABLE 9: MAGNESIUM.[1]

$K_1 = -0.008$, $K_2 = 0$, $C_2 = 0$

C, or $(kf+k)$, at 0°	$C_1 = -23 \times 10^{-6}$			$C_1 = -27 \times 10^{-6}$			$C_1 = -31 \times 10^{-6}$		
	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°
0.001	0	0.51	0.001	0.5	0.34	0.001	1	0.15	0.001
0.01	0	5.5	0.01	0.5	3.2	0.01	1	1.4	0.01
0.05	0	27.5	0.05	0.5	16.0	0.05	1	7.0	0.05

[1] The Thomson effect in magnesium was very small, and the values of δ_0 indicated for this metal are exceptionally large. Bridgman, however, describes the specimen used as of commercial quality, and thinks it was impure.

TABLE 10: SILVER.

$K_1 = -0.864$, $K_2 = 0$, $C_2 = 0$

C, or $(kf+k)$, at 0°	$C = -25 \times 10^{-6}$			$C_1 = -29 \times 10^{-6}$			$C_1 = -34 \times 10^{-6}$		
	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°
0.01	0	0.09	0.007	0.5	0.05	0.007	1	0.02	0.007
0.10	0	0.54	0.097	0.5	0.32	0.097	1	0.14	0.097
0.20	0	1.04	0.197	0.5	0.61	0.197	1	0.26	0.197
0.40	0	2.04	0.397	0.5	1.19	0.397	1	0.50	0.397

TABLE 11: TUNGSTEN.

$K_1 = -3.41$, $K_2 = 0$, $C_2 = 0$

C, or $(kf+k)$, at 0°	$C_1 = -100 \times 10^{-6}$			$C_1 = -114 \times 10^{-6}$			$C_1 = -130 \times 10^{-6}$		
	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°
0.01	0	0.05	0.000	0.5	0.03	0.000	1	0.02	0.000
0.10	0	0.16	0.090	0.5	0.10	0.089	1	0.05	0.087
0.20	0	0.29	0.190	0.5	0.17	0.189	1	0.08	0.187
0.40	0	0.55	0.390	0.5	0.32	0.389	1	0.14	0.387

Third Group: Metals for which $K_1>0$ and $K_2>0$

TABLE 12: IRON.

$K_1 = 1.78$, $K_2 = .0.0516$

C, or $(kf\div k)$, at 100°	$C_2 = 93 \times 10^{-8}$				$C_2 = 100 \times 10^{-8}$				$C_2 = 110 \times 10^{-8}$			
	q	δ_0	$C_1 \times 10^{-6}$	$(kf\div k)$ at 100°	q	δ_0	$C_1 \times 10^{-6}$	$(kf\div k)$ at 100°	q	δ_0	$C_1 \times 10^{-6}$	$(kf\div k)$ at 100°
0.001	0	.015	66	.017	.5	.010	70	.018	1	.005	77	.020
0.01	0	.019	44	.024	.5	.012	57	.026	1	.006	68	.028
0.05	0	.031	—21	.057	.5	.019	10	.061	1	.009	43	.065
0.10	0	.042	—81	.101	.5	.027	—43	.106	1	.013	9	.112
0.15	0	.052	—135	.146	.5	.034	—90	.151	1	.017	—26	.158
0.20	0	.060	—178	.192	.5	.039	—123	.198	1	.020	—52	.205
0.25	0	.068	—221	.237	.5	.045	—163	.244	1	.022	—69	.254

TABLE 13: THALLIUN.

$K_1 = 0.268$, $K_2 = 0.00336$

C, or $(kf\div k)$, at 0°	$C_2 = 6 \times 10^{-8}$				$C_2 = 6.5 \times 10^{-8}$				$C_2 = 7.1 \times 10^{-8}$			
	q	δ_0	$C_1 \times 10^{-6}$	$(kf\div k)$ at 100°	q	δ_0	$C_1 \times 10^{-6}$	$(kf\div k)$ at 100°	q	δ_0	$C_1 \times 10^{-6}$	$(kf\div k)$ at 100°
0.001	0	.020	7	.002	.5	.010	10	.003	1	.006	10	.003
0.01	0	.048	—3	.011	.5	.034	—1	.011	1	.016	5	.011
0.05	0	.130	—32	.047	.5	.090	—25	.048	1	.050	—14	.049
0.10	0	.186	—51	.096	.5	.130	—42	.096	1	.080	—31	.098
0.15	0	.227	—65	.144	.5	.160	—55	.145	1	.096	—40	.147
0.20	0	.262	—78	.193	.5	.186	—66	.194	1	.105	—45	.196
0.30	0	.324	—99	.291	.5	.220	—81	.293	1	.135	—62	.295

Fourth Group: Metals for which $K_1<0$ and $R_2>0$

TABLE 14: ALUMINUM.

$K_1 = -0.016$, $K_2 = 0.006$

C, or $(kf\div k)$, at 0°	$C_2 = 11 \times 10^{-8}$				$C_2 = 12 \times 10^{-8}$				$C_2 = 13 \times 10^{-8}$			
	q	δ_0	$C_1 \times 10^{-6}$	$(kf\div k)$ at 100°	q	δ_0	$C_1 \times 10^{-6}$	$(kf\div k)$ at 100°	q	δ_0	$C_1 \times 10^{-6}$	$(kf\div k)$ at 100°
0.001	0	.020	—1	.002	.5	.012	—1	.002	1	.006	—1	.002
0.01	0	.036	—11	.010	.5	.022	—9	.010	1	.012	—7	.011
0.05	0	.086	—41	.047	.5	.058	—37	.048	1	.032	—30	.048
0.10	0	.125	—64	.095	.5	.088	—61	.095	1	.047	—46	.097
0.20	0	.180	—97	.191	.5	.126	—91	.192	1	.068	—69	.194
0.30	0	.226	—125	.289	.5	.155	—114	.290	1	.088	—91	.292

TABLE 15: GOLD.

$K_1 = -0.934$, $K_2 = 0.001$

C, or $(kf+k)$, at 0°	$C_2 = 1.8 \times 10^{-8}$				$C_2 = 2.0 \times 10^{-8}$				$C_2 = 2.1 \times 10^{-8}$			
	q	δ_0	$C_1 \times 10^{-8}$	$(kf+k)$ at 100°	q	δ_0	$C_1 \times 10^{-8}$	$(kf+k)$ at 100°	q	δ_0	$C_1 \times 10^{-8}$	$(kf+k)$ at 100°
0.001	0	.032	—29	—	.5	.024	—33	—	1	.011	—38	—
0.01	0	.051	—31	.007	.5	.040	—35	.007	1	.018	—39	.006
0.05	0	.122	—38	.046	.5	.096	—42	.046	1	.050	—44	.046
0.10	0	.188	—45	.096	.5	.158	—50	.095	1	.083	—50	.095
0.20	0	.292	—56	.195	.5	.256	—62	.194	1	.136	—58	.194
0.25	0	.334	—60	.244	.5	.294	—77	.293	1	.158	—64	.244

TABLE 16: MOLYBDENUM.

$K_1 = -4.334$, $K_2 = 0.015$

C, or $(kf+k)$, at 0°	$C_2 = 27 \times 10^{-8}$				$C_2 = 29 \times 10^{-8}$				$C_2 = 32 \times 10^{-8}$			
	q	δ_0	$C_1 \times 10^{-8}$	$(kf+k)$ at 100°	q	δ_0	$C_1 \times 10^{-8}$	$(kf+k)$ at 100°	q	δ_0	$C_1 \times 10^{-8}$	$(kf+k)$ at 100°
0.001	0	.025	—138		.5	.017	—155		1	.008	—175	
0.01	0	.030	—146		.5	.020	—161		1	.010	—180	
0.05	0	.047	—173	.035	.5	.030	—180	.035	1	.015	—192	.044
0.10	0	.064	—197	.083	.5	.042	—204	.083	1	.020	—205	.083
0.20	0	.094	—220	.181	.5	.061	—240	.179	1	.030	—230	.180

Comments

Lead does not appear in these tables, for the reason that σ for this metal is so small that we have no formula for it. Zero value for σ could be accounted for, according to the principles of this paper, in either of two ways:

First, as equation (8) shows, $(k_f \div k)$ might be zero.

Second, if in equation (8) we make $(kf \div k)$ a constant, of whatever value,[4] and is zero if $n \propto T^{1.5}$

These tables, like the experimental values of σ by means of which they are made, have no great pretension to accuracy. For example, it is doubtful whether much confidence can be placed in the small value of K_2, very nearly 0.001, found by Bridgman for gold. Yet the effect of neglecting this value is considerable. Thus, if we call $K_2 = 0$ for gold, we get

TABLE 15 *bis*: GOLD.

$K_1 = -0.934$, K_2 taken to be 0, and $C_3 = 0$

C, or $(k_f \div k)$, at 0°	$C_1 = -27 \times 10^{-6}$			$C_1 = -31 \times 10^{-6}$			$C = -36 \times 10^{-6}$		
	q	δ_0	$(k_f \div k)$ at 100°	q	δ_0	$(k_f \div k)$ at 100°	q	δ_0	$(k_f \div k)$ at 100°
0.01	0	0.08	0.007	0.5	0.05	0.007	1	0.02	0.006
0.10	0	0.51	0.097	0.5	0.30	0.097	1	0.13	0.096
0.20	0	0.99	0.197	0.5	0.58	0.197	1	0.25	0.196
0.40	0	1.95	0.397	0.5	1.14	0.397	1	0.49	0.396

Comparison of this with table 15 shows that ignoring the value of K_2, small as it is, makes a good deal of difference in the values of δ_0, though comparative little in the values of $(k_f \div k)$ at 100°, as calculated by means of equation (10).

Uncertain as the values of $(k_f \div k)$ are, there is little room for doubt in most cases, if my theory is substantially sound, as to the direction of change of these values when the temperature is raised. Accordingly I have divided all the metals represented by the preceding tables into two groups, in the first of which $(k_f \div k)$ is greater at 100° than at 0°, while in the second group the ratio in question is greater at 0°.

For most of these metals Bridgman has determined the pressure-coefficient of resistance at 0° and 100°. For magnesium the value at 0° only was found; for bismuth the highest temperature used was 75°. For all of the metals here considered except bismuth the coefficient in question is negative—that is, the resistance decreases with increase of pressure—but for bismuth it is positive. I shall make use of what Bridgman calls the *average* pressure-coefficient, the average value of the coefficient through a range of pressure from 0 kgm. to 12000 kgm. per square centimeter. I shall let π_0 represent the value of this coefficient at 0°, and π_{100} the value at 100°.

In accordance with what has been said in the opening paragraphs of this paper we should, other things being equal, expect π to decrease, numerically, with increase of $(k_f \div k)$ in metals for which π is negative, and to increase with increase of $(k_f \div k)$ in metals for which π is positive. Accordingly we might expect $(\pi_{100} - \pi_0) \div \pi_0$ to be, in general, a negative quantity for metals in which $(k_f \div k)$ increases with rise of temperature, and a positive quantity for metals in which $(k_f \div k)$ decreases with rise of temperature—bismuth of course requiring exceptional consideration. The table given below enables us to test the validity of this expectation.

TABLE 17

(A)

METALS IN WHICH $(k_f \div k)$ INCREASES WITH RISE OF TEMPERATURE

	π_0	π_{100}	$(\pi_{100} - \pi_0) \div \pi_0$
Co	$-0.0_{6}873$	$-0.0_{6}726$	−16.9%
Ni	$-0.0_{5}147$	$-0.0_{5}158$	+7.5%
Pd	$-0.0_{5}190$	$-0.0_{5}186$	−2.1%
Pt	$-0.0_{5}187$	$-0.0_{5}184$	−1.6%
St	$-0.0_{5}920$	$-0.0_{5}951$	+3.4%
Zn	$-0.0_{5}470$	$-0.0_{5}454$	−3.4%
		Mean	−2.2%

(B)

METALS IN WHICH $(k_f \div k)$ DECREASES WITH INCREASE OF TEMPERATURE

	π_0	π_{100}	$(\pi_{100} - \pi_0) \div \pi_0$
Ag	$-0.0_{5}333$	$-0.0_{5}336$	+0.9%
Al	$-0.0_{5}382$	$-0.0_{5}377$	−1.3%
An	$-0.0_{5}287$	$-0.0_{5}292$	+1.7%
Bi[1]	$+0.0_{4}223$	$+0.0_{4}202$	+9.4%
Cd	$-0.0_{5}894$	$-0.0_{5}927$	+3.7%
Cu	$-0.0_{5}183$	$-0.0_{5}177$	−3.3%
Mg[2]	$-0.0_{5}55$		
Mo	$-0.0_{5}129$	$-0.0_{5}127$	−1.6%
Ta	$-0.0_{4}115$	$-0.0_{4}123$	+7.0%
W	$-0.0_{5}123$	$-0.0_{5}126$	+2.4%
		Mean	+2.1%

[1] For bismuth, because π is positive and the highest temperature for π was 75°, we use $(\pi_0 - \pi_{75}) \div \pi_0$.

[2] Bridgman did not find π_{100} for magnesium.

Iron does not appear in this table, for the reason that, as table 12 shows, it should go into Section (*A*) under some conditions, but into Section (*B*) under other conditions. Its value of $(\pi_{100} - \pi_0) \div \pi_0$ is about 4%. This value, if put into Section (*A*), would make the mean value there −1.3%; put into Section (*B*), it would make the mean there + 2.3%. If $(k_f \div k)$ for iron is greater than 10% at 0°*C*., which seems likely, iron belongs to Section (*B*).

Thallium and aluminium should go into Section (*A*) if $(k_f \div k)$ in them is less than 1% at 0°*C;* but this is improbable.

It is to be observed that for each section of table 17 the average value of $(\pi_{100} - \pi_0) \div \pi_0$ comes out with the sign it should have according to the predictions of the dual theory of electric conduction, as used in this paper, a *minus* sign for Section (*A*) and a *plus* sign for Section (*B*). This can hardly be pure accident.

Under Hypothesis (*B*)

All of the equations and all the tables of this paper, thus far, are based upon or are consistent with "hypothesis (*A*)." If hypothesis (*B*) is adopted instead, equation (8) is changed in the manner already described and the result in equations (13) to (19) is to replace 1.5 by 2, 4 by 4.5, and 6.5 by 7, in the parenthesis $(1.5 - q)$, $(4 - q)$ and $(6.5 - q)$.

The resulting tables, for cobalt and bismuth, the only metals for which the calculation has been made, are given in Tables 18 and 19 (below).

By comparison of these two tables with the corresponding ones obtained by the use of hypothesis (*A*) we see that no radical difference in the re-

sults comes from the substitution of one hypothesis for the other. Hypothesis (*B*) seems to give, other things being equal, somewhat larger values of δ_0 than hypothesis (*A*).

TABLE 18: COBALT.

$K = 7.8$, $K_2 = 0$, $C_2 = 0$

C, or $(kf+k)$, at 0°	$C_1 = 202 \times 10^{-4}$			$C_1 = 230 \times 10^{-4}$			$C_1 = 260 \times 10^{-4}$			$C_1 = 303 \times 10^{-4}$		
	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°
0.01	0	.038	.030	0.5	.029	.033		.020	.036	1.5	.010	.040
0.05	0	.004	.070	0.5	.007	.073		.007	.076	1.5	.005	.080
0.10	0	—		0.5	—			—		1.5	—	

C, or $(kf+k)$, at 0°	$C_1 = 455 \times 10^{-4}$			$C_1 = 607 \times 10^{-4}$		
	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°
0.10	2.5	—		3	—	
0.15	2.5	0.002	0.196	3	—	
0.20	2.5	0.007	0.246	3	0.005	0.261
0.30	2.5	0.016	0.346	3	0.019	0.361
0.40	2.5	0.026	0.446	3	0.032	0.461

TABLE 19: BISMUTH.

$K_1 = -3.2$, $K_2 = 0$, $C_2 = 0$

C, or $(kf+k)$, at 0°	$C_1 = -83 \times 10^{-4}$			$C_1 = -93 \times 10^{-4}$			$C_1 = -106 \times 10^{-4}$		
	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°	q	δ_0	$(kf+k)$ at 100°
0.01	0	0.07	0.002	0.5	0.05	0.001	1.5	0.03	0.000
0.10	0	0.25	0.092	0.5	0.17	0.091	1.5	0.10	0.089
0.20	0	0.46	0.192	0.5	0.32	0.191	1.5	0.18	0.189
0.40	0	0.87	0.392	0.5	0.58	0.391	1.5	0.34	0.389
0.60	0	1.28	0.592	0.5	0.84	0.591	1.5	0.50	0.589

In a paper already well advanced I shall undertake to show how far the data obtained in the present paper enable us to go in the way of explaining thermal conduction in the metals here dealt with.

[1] Eq. (7) is obtained from eq. (1) of my paper in the *Proceedings of the National Academy of Sciences* for April, 1918, by substituting for μ from eq. (2) of that paper (which equation should have n instead of m) and then making obvious changes.

[2] *Proceedings of the Amer. Acad. of Sciences*, Vol. **53**, No. 4, 1918, pp. 269–386. These values are not regarded by Bridgman as accurate, having been obtained as second derivatives of e. m. f. values, the quantities measured; but taken as a whole they seem to be the best available data for the present purpose.

[3] This assumes that λ'_0 is constant.

[4] When (k_f+k) is infinite, we have the conditions discussed in my paper "Thermoelectric Diagrams on the P-V Plane," *Proc. Amer. Acad., Boston*, February, 1918.

NOTE ON GEOMETRICAL PRODUCTS

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Geometrical products may be divided into two general classes: those definable in a space of any number of dimensions and those definable only in a space of a definite number of dimensions. To the first class belong the progressive (outer) product of Grassmann and the inner products of Grassmann, Gibbs[1] and Lewis.[2] To the second class belong the regressive product of Grassmann and the cross product of Gibbs (as defined in the Gibbs-Wilson Vector Analysis) and the quaternion multiplications. In this paper we show that there is a series of geometrical products independent of dimensions that may be in a sense considered intermediate between the progressive product and the inner product.

In terms of the units[3] the progressive product is expressed by identities of the form $k_{12}k_{34} = k_{1234}$, the product containing all the units multiplied together. If there are any common units as in $k_{12}k_{14} = 0$, the product is zero. The inner product of Lewis is expressed by identities of the form $k_3.k_{123} = k_{12}$, the common units being cancelled (with certain conventions as to algebraic sign). Unless one factor is entirely contained in the other the result is zero.

If π_1 and π_2 are products of units whose first m factors are the same and in the same order we define as the product $[\pi_1\pi_2]_m$ of index m the result of cancelling those m common units and taking the outer product of the remaining units in the order written. Thus

$$[k_{12}k_{13}]_1 = k_{23}, \quad [k_{123}k_{124}]_2 = k_{34}.$$

If π_1 and π_2 have m units in common but these are not the first m units, π_1 and π_2 are rewritten (with change of sign if necessary as required by the outer multiplication) in such a way that the first m factors are the same, and in the same order, the product is then formed as above, negative signs introduced being included.[4] Thus

$$[k_{12}k_{23}]_1 = -k_{13}, \quad [k_{234}k_{134}]_2 = k_{21} = -k_{12}.$$

If there are either more or less than m units in common, the product is zero. Thus

$$[k_{123}k_{124}]_1 = k[_{123}k_{124}]_3 = 0.$$

The inner product is obtained when the index is equal to the dimension of the lowest factor. Thus

$$[k_1k_{12}]_1 = k_1.k_{12} = -k_2.$$

The outer product occurs when the index is zero. Thus $[k_{12}k_{34}]_0 = k_{1234}$.

To obtain the product of M and N, two homogeneous linear functions of units, they are multiplied distributively, the product of units being found

as above. We shall now show that the product of two such quantities is invariant, i. e., that it is independent of the system of units in terms of which it is expressed. We do this by showing that $[MN]_m$ can be expressed in terms of invariant operations on M, N and the identical transformation[5] (dyadic) I.

If $k_1, k_2, \ldots k_n$ are the unit one-vectors, the 1, 1 idemfactor is

$$I_1 = k_1k_1 + k_2k_2 + \ldots + k_nk_n.$$

Using the dot to express inner multiplication we then have

$$k_{12}.I_1 = k_1k_2 - k_2k_1 = -I_1.k_{12},$$
$$k_{123}.I_1 = k_{23}k_1 - k_{13}k_2 + k_{12}k_3,$$
$$I_1.k_{123} = k_1k_{23} - k_2k_{13} + k_3k_{12},$$

and similar expressions for products with units of higher order. Using these values we find

$$[k_{12}k_{13}]_1 = [(k_{12}.I_1).(I_1.k_{13})]_0,$$

where the subscript 0 indicates that the antecedent and consequent of each dyad in the brackets are multipled according to the (outer) multiplication of index zero and the sum of terms then taken. Similarly we find

$$[k_{123}k_{145}]_1 = [(k_{123}.I_1).(I_1.k_{145})]_0,$$

and generally

$$[\pi_1\pi_2]_1 = [(\pi_1.I_1).(I_1.\pi_2)]_0$$

where π_1 and π_2 are any product of units. Since the operations are all distributive we have then

$$[MN]_1 = [(M.I_1).(I_1.N)_0].$$

Since the dot multiplication, the outer (zero) multiplication and I_1 are invariant it follows that $[MN]_1$ is invariant.

For the product of index 2, we use the 2, 2 idemfactor $I_2 = \Sigma k_{ij}k_{ij}$ with $i<j$. We then find

$$[k_{123}k_{124}]_2 = [(k_{123}.I_2).(I_2.k_{124})]_0$$

and generally

$$[\pi_1\pi_2]_2 = [(\pi_1.I_2).(I_2.\pi_2)]_0,$$

or since the operations are distributive

$$[MN]_2 = [(M.I_2).(I_2.N)]_0$$

which shows the invariance as before.

In general the product of index m is

$$[MN]_m = [(M.I_m).(I_m.N)]_0$$

where

$$I_m = \Sigma k_{12\ldots m}\, k_{12\ldots m}.$$

Let M_p be a simple space of dimension p and N_q a simple space of dimension q.

Theorem I. In order that $[M_pN_q]_m = 0$ it is necessary and sufficient that

$$[M_p(R_m.N_q)]_0 = 0. \qquad (1)$$

where R_m is an arbitrary simple space of dimension m in M_p.

To prove this take the units $k_1, k_2, \ldots k_n$ so that $k_1, k_2, \ldots k_p$ lie in M_p and $R_m = k_{12\ldots m}$. Then

$$[M_pN_q]_m = \Sigma[(M_p.k_{12\ldots m})(k_{12\ldots m}.N_q)]. \quad (2)$$

In the summation terms containing $k_{p+1}, k_{p+2}, \ldots k_n$ may be omitted since the product with M_p would be zero. The outer product of R_m and (2) is

$$[R_m(M_pN_q)_m]_0 = [R_m(M_p.k_{12\ldots m})(k_{12\ldots m}.N_q)]_0 = 0, \quad (3)$$

all other terms vanishing because they contain repeated factors. Now

$$[R_m(M_p.k_{12\ldots m})]_0 = [k_{12\ldots m}(M_p.k_{12\ldots m})]_0 = M_p.$$

Hence (3) is equivalent to

$$[M_p(R_m.N_q)]_0 = 0$$

which was to be proved.

Theorem II. A necessary and sufficient condition that

$$[M_pN_q]_m = 0 \quad (4)$$

is that either M_p and N_q intersect in a space of dimensions $\geqq m+1$ or M_p contain a space of dimension $p - m + 1$ completely perpendicular to N_q and N_q contain a space of dimensions $q - m + 1$ completely perpendicular to M_p.

If (4) is satisfied, by (1), either $R_m.N_q = 0$ or the outer product of M_p and $R_m.N_q$ is zero. In the first case R_m contains a perpendicular to N_q and consequently M_p must contain a space perpendicular to N_q which is cut by every R_m. For this it is necessary and sufficient that M_p contain a space of dimensions $p - m + 1$ perpendicular to N_q. In the second case the intersection of M_p and N_q must contain a vector perpendicular to any R_m in M_p. The intersection must then be of such a dimension that it cuts all spaces in M_p of dimension $p - m$. It must then be of dimension $m + 1$.

Theorem III. If

$$[M_pN_q]_m = 0, [M_pN_q]_{m+1} \neq 0, \quad (5)$$

M_p and N_q will intersect in a space of dimension $m + 1$ and so all the products of M_p and N_q of index equal to or less than m will be zero; and, if

$$[M_pN_q]_n = 0, [M_pN_q]_{n-1} \neq 0, \quad (6)$$

M_p will contain a space of dimension $p - m + 1$ completely perpendicular to N_q and so all products of index equal to or greater than n will be zero.

In fact if relations (5) are satisfied, the vanishing of $[M_pN_q]_m$ must be due to M_p and N_q intersecting in a space of dimension $m + 1$; for if there were in M_p a space of dimension $p - m + 1$ perpendicular to N_q, the product $[M_pN_q]_{m+1}$ would also vanish. Hence, in this case all the products of index less than m will also vanish. If (6) is satisfied it must be due to M_p containing a space of dimension $p - m + 1$ completely perpendicular to N_q, for if M_p and N_q intersect in a space of order $m + 1$

the product $[M_pN_q]_{n-1}$ will also be zero, hence, all the products of index less than n will vanish.

It is clear from the theorem just proved that all the products $[M_pN_q]_r$ for which $m<r<n$ will be different from zero and all the products for which $r \leqq m$ or $r \geqq n$ will be zero.

Theorem IV. If equations (5) are satisfied, $[M_pN_q]_{m+1}$ is the outer product of the parts of M_pN_q completely perpendicular to the intersection.

In this case M_p and N_q intersect in a space of $m+1$ dimensions. Let $k_1, k_2, \ldots k_{m+1}$ be unit vectors in that space. Then

$$M_p = k_1k_2\ldots k_{m+1}P$$

and

$$N = k_1k_2\ldots k_{m+1}Q$$

where P is the part of M_p perpendicular to $k_1k_1\ldots k_{m+1}$ and Q is the part of N_q completely perpendicular to $k_1k_1\ldots k_{m+1}$. Hence

$$[M_pN_q]_{m+1} = [PQ]_0$$

which proves the theorem.

Theorem V. If (6) is satisfied $[M_pN_q]_{n-1}$ is the outer product of the part of M_p perpendicular N_q and the part of N_q perpendicular to M_p.

Let $l_1, l_2, \ldots . l_{p-n+1}$ be the units in M_p perpendicular to N_q and $j_1, j_2, \ldots . j_{q-n+1}$ the units in N_q perpendicular to M_p. These units are then perpendicular to each other and

$$M_p = Pl_1l_2\ldots . l_{p-n+1}$$
$$N_q = Qj_1j_2\ldots . j_{q-n+1}$$

where P (of dimension $n-1$) is the part of M_p perpendicular to $l_1l_2\ldots l_{p-n+1}$ and Q is the part of N_q perpendicular to $j_1j_2\ldots . j_{q-n+1}$. Since the units $l_1l_2\ldots . l_{p-n+1}, j_1j_2\ldots . j_{q-n+1}$ are all distinct

$$[M_pN_q]_{n-1} = [PQ]_{n-1}[l_1l_2\ldots . l_{p-n+1}\, j_1j_2\ldots . j_{q-n+1}].$$

Since $[PQ]_{n+1}$ is a number this proves the theorem.

[1] Gibbs, Wilson, *Vector Analysis*.

[2] *Proc. Amer. Acad. Arts Sci.*, **46**, 1910 (165–181).

[3] The products here discussed occur as partial products in the multiplication of generalized quaternions. From this point of view they have been treated by Clifford (*Amer. J. Math.*, 1), Joly (*Proc. Royal Irish Acad.*, 5, (3)) McAuley (*Proc. R. Soc. Edinburgh*, 28), Shaw (*Bull. Int. Assoc. Adv. Quaternions*, 1913, (24–27)), and others. In victor algebras of the Grassmann type, for example in the usual victor theory of relativity, they have, however, such a different appearance that we have presented them here from that point of view.

[4] In case the index is equal to the order of one of the factors the sign is taken as in the Lewis inner product.

[5] We used this as the definition of the product of index 1 (star product) in a paper "Rotations in Space of Even Dimensions" to appear in *Proc. Amer. Acad. Arts Sci.*